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The Partial L-Moment of the Four Kappa Distribution

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Abstract

Statistical analysis of extreme events such as flood events is often carried out to predict large return period events. The behaviour of extreme events not only involves heavy-tailed distributions but also skewed distributions, similar to the four-parameter Kappa distribution (K4D). In general, this covers many extreme distributions such as the generalized logistic distribution (GLD), the generalized extreme value distribution (GEV), the generalized Pareto distribution (GPD), and so on. To utilize these distributions, we have to estimate parameters accurately. There are many parameter estimation methods, for example, Method of Moments, Maximum Likelihood Estimator, L-Moments, or partial L-Moments. Nowadays, no researchers have applied the partial L-Moments method to estimate the parameters of K4D. Therefore, the objective of this paper is to derive the partial L-Moments (PL-Moments) for K4D, namely the PL-Moments of the K4D in order to estimate hydrological extremes from censored data. The findings of this paper are formulas of parameter estimation for K4D based on the PL-Moments approach. We have derived the Partial Probability-Weighted Moments (PPWMs) of the K4D (β_r') and derive the estimation of parameters when separated by shape parameters (k, h) conditions i.e., case $k > -1$ and $h > 0$, case $k > -1$ and $h = 0$ and case $-1 < k < -\frac{1}{h}$ and $h < 0$. Finally, we expect that the parameter estimate for K4D from this formula will help to make accurate forecasts.

Keywords:

Four Kappa Distribution;

L-Moments;

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1- Introduction

At present, extreme events often occur, such as the phenomenon of severe torrential rain causing flash flooding, severe snow both in-season and off-season, wildfires, heat waves, or sudden changes of temperature. Such extreme events cause severe disasters affecting humans, living creatures, and property. To solve these extreme problems, accurate forecasting of extreme events is helpful for planning and being able to cope with disasters. Accordingly, statistical analysis is an effective forecasting method that is widely used.

When considering in depth the behaviours of extreme events, the data are in a heavy-tailed pattern or heavy-tailed distribution, especially the right heavy-tailed distribution known as the upper heavy-tailed distribution. For the upper heavy-tailed distribution, the four-parameter Kappa distribution consists of 1 location parameter (ξ), 1 scale parameter (α), and 2 shape parameters (k, h). If the value of the shape parameter changes, in the four-parameter Kappa distribution, the change of the shape distribution will be as follows. It becomes the generalized logistic distribution (GLD) in the case of $k \neq 0, h = -1$, the generalized extreme value distribution (GEV) in the case of $k \neq 0, h = 0$, the generalized Gumbel distribution (GGD) in the case of $k = 0, h \neq 0$, and the generalized Pareto distribution (GPD) in the case of $k \neq 0$, $h = 1$. Each distribution mentioned above is in a heavy-tailed pattern in accordance with the disaster.

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The statistical analysis for parameter estimation is one effective forecasting method. Many methods for parameter estimation are available such as Method of Moments, Maximum Likelihood Estimator, L-Moments, partial L-Moments, and so on. Method of Moments is the oldest method, invented by Pearson [1]. The estimators obtained from this method are easily calculated so it is suitable for estimating a parameter with a small number of parameters. In general, the obtained estimator is consistent with little bias, but most obtained estimators are low-effective. In 1912, Fisher [2] developed the method of Maximum Likelihood Estimator which is a simple and widely-used estimation. Generally, the obtained estimators are effective and consistent. This estimation method is suitable for Gaussian asymptotic distributions. However, this method contains limitations that the obtained estimators are low-effective if the data are small, and the obtained estimation values contain high bias and variance.

Many parameter estimation methods have been developed such as L-Moments proposed by Hosking [3]. This method calculates estimators on the basis of linear function of expected values of order statistics. In comparing between the estimating results of small-sized data, the parameter-estimated values obtained from L-Moments are better than the values obtained from the method of Maximum Likelihood Estimator [4]. Therefore, this method of parameter estimation is widely applied in various disciplines such as engineering, quality control, meteorology, and hydrology [5, 6]. However, when using the data for estimating parameters and forecasting recurrence periods such as 50-year recurrence or 100-year recurrence, the forecasting values are obtained from the estimation of all data. The main forecasting objective is to plan for coping with disaster. Therefore, attention should be paid to the weight of data. Greenwood et al. [7] propose the method of Probability Weighted Moments (PWMs) to solve such problems. However, disaster events need to be specifically studied with extreme data. Therefore, the data are screened to use only the data with effects on extreme events for the analysis. These selected data are called censored samples and this parameter estimation method is called Partial Probability Weighted Moments (PPWMs) [8]. In addition, Wang [8] proposes the use of linear combination for PWMs as a new estimation method, known as partial L-Moments (PL-Moments). When comparing the results of parameter estimation of extreme events with the GEV distribution between PWMs and PPWMs, it is found that PPWMs gives better results of parameter estimation than PWMs. Moreover, it is also found that PL-Moments gives better results of parameter estimation than L-Moments [9, 10]. There are a variety of research studies about methods of parameter estimations with the four-parameter Kappa distribution, such as Seenoi et al. [11] who proposed Bayesian parameter estimation with the four-parameter Kappa distribution. Ibrahim [12] studied parameter estimation in light of the L-Moments with the four-parameter Kappa distribution. Papukdee et al. [13] proposed a penalized likelihood approach for the four-parameter Kappa distribution. Shin & Park [14] derived parameter estimation using r-largest order statistics with the four-parameter Kappa distribution. Aranda [15] studied frequency analysis using L-Moments with the Kappa distribution.

Nowadays, many events are extreme events, for example, floods, droughts, pollution and so forth, that may be regarded as special cases of a four-parameter Kappa distribution (K4D). There are some distributions which are special cases of K4D such as the generalized logistic distribution (GLD), the generalized extreme value distribution (GEV), the generalized Pareto distribution (GPD), and so on. The K4D is widely and flexibly applicable to the data including not only extreme values but also skewed data. Therefore, the objectives of this research are to derive the formulas using the PL-Moments approach to estimate the parameters of the K4D, namely PL-Moments of the K4D.

2- Four Parameter Kappa Distribution

In 1994, Hosking [16] presented the new distribution, namely the four parameter Kappa distribution (K4D) which exhibits heavy-tailed behaviour. There are four parameters in K4D: a location parameter (ξ), a scale parameter (α), and two shape parameters (k, h). Moreover, K4D is a parent distribution of the generalized extreme value distribution (GEV) if $k \neq 0, h = 0$, the generalized logistic distribution (GLD) if $k \neq 0, h = -1$, the generalized Gumbel distribution (GGD) if $k = 0, h \neq 0$, and the generalized Pareto distribution (GPD) if $k \neq 0, h = 1$. The cumulative distribution function (cdf) of the K4D is [16]:

$$F(x) = \begin{cases} \left\{1 - h\left[1 - \frac{k(x-\xi)}{\alpha}^{\frac{1}{k}}\right]\right\}^{\frac{1}{h}}; k \neq 0, h \neq 0, -\infty < x < \infty \\ exp\left\{-\left[1 - \frac{k(x-\xi)}{\alpha}\right]^{\frac{1}{k}}\right\}; k \neq 0, h \neq 0, -\infty < x < \infty \\ \left\{1 - hexp\left[-\left(\frac{x-\xi}{\alpha}\right)\right]\right\}^{\frac{1}{h}}; k = 0, h = 0, -\infty < x < \infty \\ exp\left\{-exp\left[-\left(\frac{x-\xi}{\alpha}\right)\right]\right\}; k = 0, h = 0, -\infty < x < \infty \end{cases}, \tag{1}$$

The probability density function (pdf) of the K4D is:

$$f(x) = \begin{cases} \frac{1}{\alpha}\left[1 - \frac{k(x-\xi)}{\alpha}\right]^{\frac{1}{k}-1} [F(x)]^{1-h}; k \neq 0, h \neq 0, -\infty < x < \infty \\ \frac{1}{\alpha}\left[1 - \frac{k(x-\xi)}{\alpha}\right]^{\frac{1}{k}-1} F(x); k \neq 0, h = 0, -\infty < x < \infty \\ \frac{1}{\alpha}\left[\exp\left(-\frac{(x-\xi)}{\alpha}\right)\right] [F(x)]^{1-h}; k = 0, h \neq 0, -\infty < x < \infty \\ \frac{1}{\alpha}\left[\exp\left(-\frac{(x-\xi)}{\alpha}\right)\right] F(x); k = 0, h = 0, -\infty < x < \infty \end{cases}, \tag{2}$$

The quantile function (qf) of the K4D is:

$$x(F) = \begin{cases} \xi + \frac{\alpha}{k}\left[1 - \left(\frac{1-F^h}{h}\right)^2\right]; k \neq 0, h \neq 0 \\ \xi + \frac{\alpha}{k}[1 - (-lnF)^k]; k \neq 0, h = 0 \\ \xi - \alpha ln\left(\frac{1-F^h}{h}\right); k = 0, h \neq 0 \\ \xi - \alpha ln(-lnF); k = 0, h = 0 \end{cases}, \tag{3}$$

Figure 1 presents some of the possible shapes of $f(x)$ for the K4D with different values of k and h, where $\xi = 0$ and $\alpha = 1$.

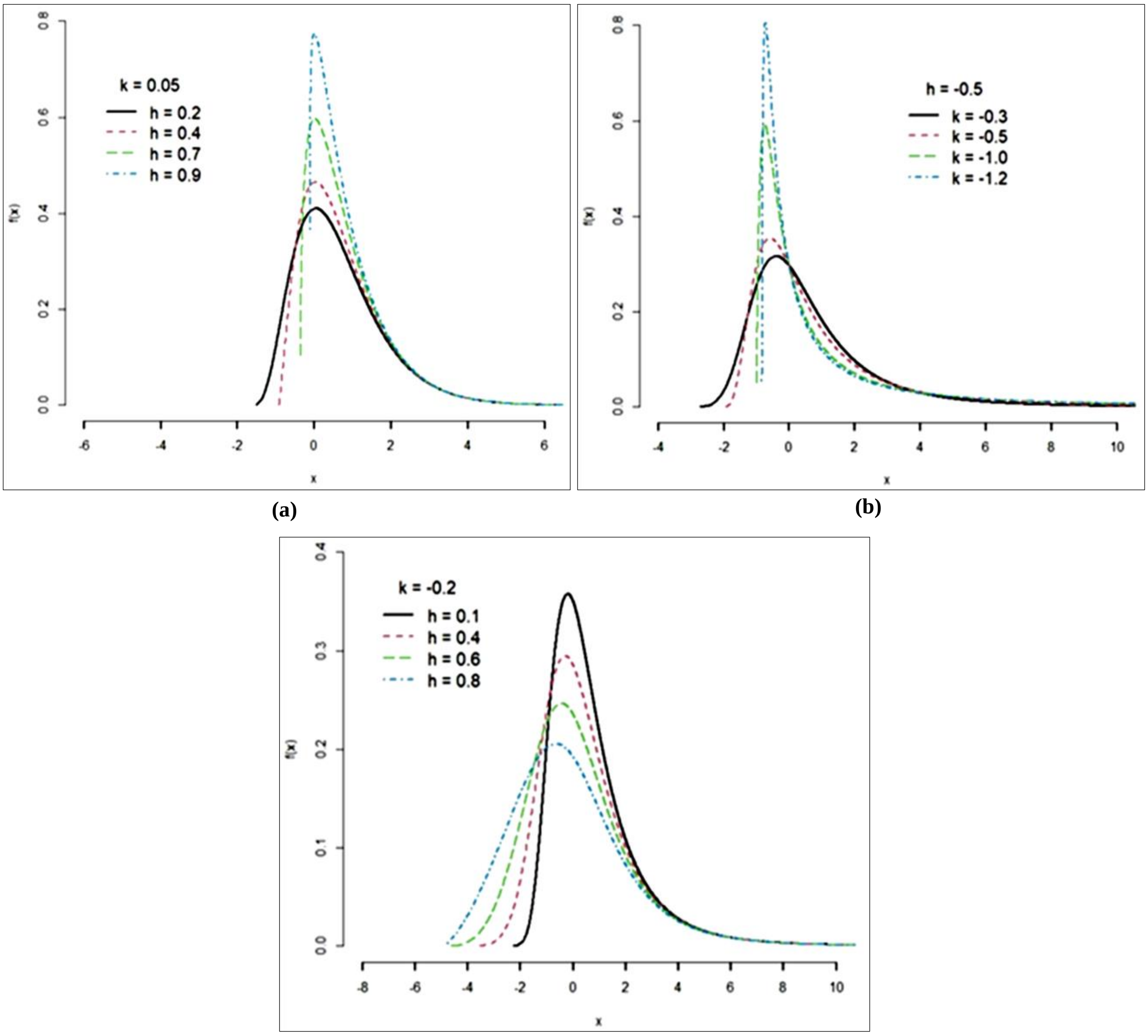

Figure 1. Shape of the pdf of the K4D plotted for various shape parameters

Apart from the above noted specific cases, the K4D is an exponential distribution if $k = 0, h = 1$, a Gumbel distribution if $k = 0, h = 0$, a logistic distribution if $k = 0, h = -1$, and a uniform distribution if $k = 1, h = 1$ [16]. Interested readers are referred to Hosking [16] for more details of the derivation, properties of this distribution, shapes of the pdf, and relations to the three parameter Kappa and Burr distributions. Winchester [17] has used artificial data using K4D.

For the estimation of parameters of the K4D, Hosking [16] used the L-Moments method, which gives more reliable performance than the Method of Moments and is usually computationally more tractable than the MLE. L-Moment estimates of K4D have been important for regional frequency analysis [18]. Winchester [17] compared the performance of MLE and L-Moments in this distribution. Park and Park [19] presented the method of Maximum Likelihood to estimate the parameters of K4D, while Park and Yoon Kim [20] investigated the Fisher information matrix for K4D. Moreover, Park et al. [21] studied a three-parameter Kappa distribution, including the comparison of MLE and L-Moment estimates. Murshed et al. [22] investigated the LH-Moment method for K4D.

3- Method of Partial L-Moments

The Probability-Weighted Moments (PWMs) is the precursor of L-Moments [7]. The PWMs can be defined as β_r which is $p = 1$ and $s = 0$ of $M_{p,r,s}$ where $M_{p,r,s} = \int_0^1 [x(F)]^p F^r (1-F)^s dF$.

$$\text{So, } \beta_r = \int_0^1 x(F) F^r dF \, ; r = 0, 1, 2, \ldots \tag{4}$$

Wang [9] has extended PWMs to Partial Probability-Weighted Moments (PPWMs) for analysing censored data. He defined the formula of the PPWMs as:

$$\beta_r^{/} = \frac{\int_{F_0}^1 x(F) F^r dF}{1 - F_0^{r+1}} ; r = 0, 1, 2, \ldots \tag{5}$$

where $0 \leq F_0 = F(x_0) \leq 1$, when $F_0 = 0$, the $\beta_r^{/}$ becomes β_r and x_0 being the censoring threshold.

In terms of PPWMs, the first four PL-Moments ($\lambda_1^{/}, \lambda_2^{/}, \lambda_3^{/}$ and $\lambda_4^{/}$) have similar properties to the first four L-Moments [18]:

the measure of the location: $$\lambda_1^{/} = \beta_0^{/}, \tag{6}$$

the characteristics of the spread: $$\lambda_2^{/} = 2\beta_1^{/} - \beta_0^{/}, \tag{7}$$

the reflects of the asymmetry of the upper part: $$\lambda_3^{/} = 6\beta_2^{/} - 6\beta_1^{/} + \beta_0^{/}, \tag{8}$$

the reflects of the peak of the upper part: $$\lambda_4^{/} = 20\beta_3^{/} - 30\beta_2^{/} + 12\beta_1^{/} - \beta_0^{/}, \tag{9}$$

and PL-Moments ratio such as PL-coefficient of variation, PL-skewness and PL-kurtosis are written as:

PL-coefficient of variation: $$\tau_2^{/} = \frac{\lambda_2^{/}}{\lambda_1^{/}}, \tag{10}$$

PL-skewness: $$\tau_3^{/} = \frac{\lambda_3^{/}}{\lambda_2^{/}}, \tag{11}$$

PL-kurtosis: $$\tau_4^{/} = \frac{\lambda_4^{/}}{\lambda_2^{/}}. \tag{12}$$

The procedures for processing this research are shown in Figure 2.

In the next Section, the above properties were used to derive PL-Moments of the K4D.

4- Results

4-1- PL-Moments of the K4D

Theorem 1: The L-Moments λ_r, $r = 1,2,3, \ldots$ of a real-valued random variable X exist if and only if X has a finite mean.

Therefore, the mean of the K4D exists if $k > -1$and $h \geq 0$ or $-1 < k < -\frac{1}{h}$and $h < 0$ [23]. It can be divided into 6 cases as follows:

when $k \neq 0$, there are 3 cases, case 1 $k > -1$and $h > 0$,

case 2 $k > -1$and $h = 0$,

case 3 $-1 < k < -\frac{1}{h}$ and $h < 0$, when $k = 0$, there are 3 cases, case 4 $h > 0$,

case 5 $h = 0$,

case 6 $h < 0$.

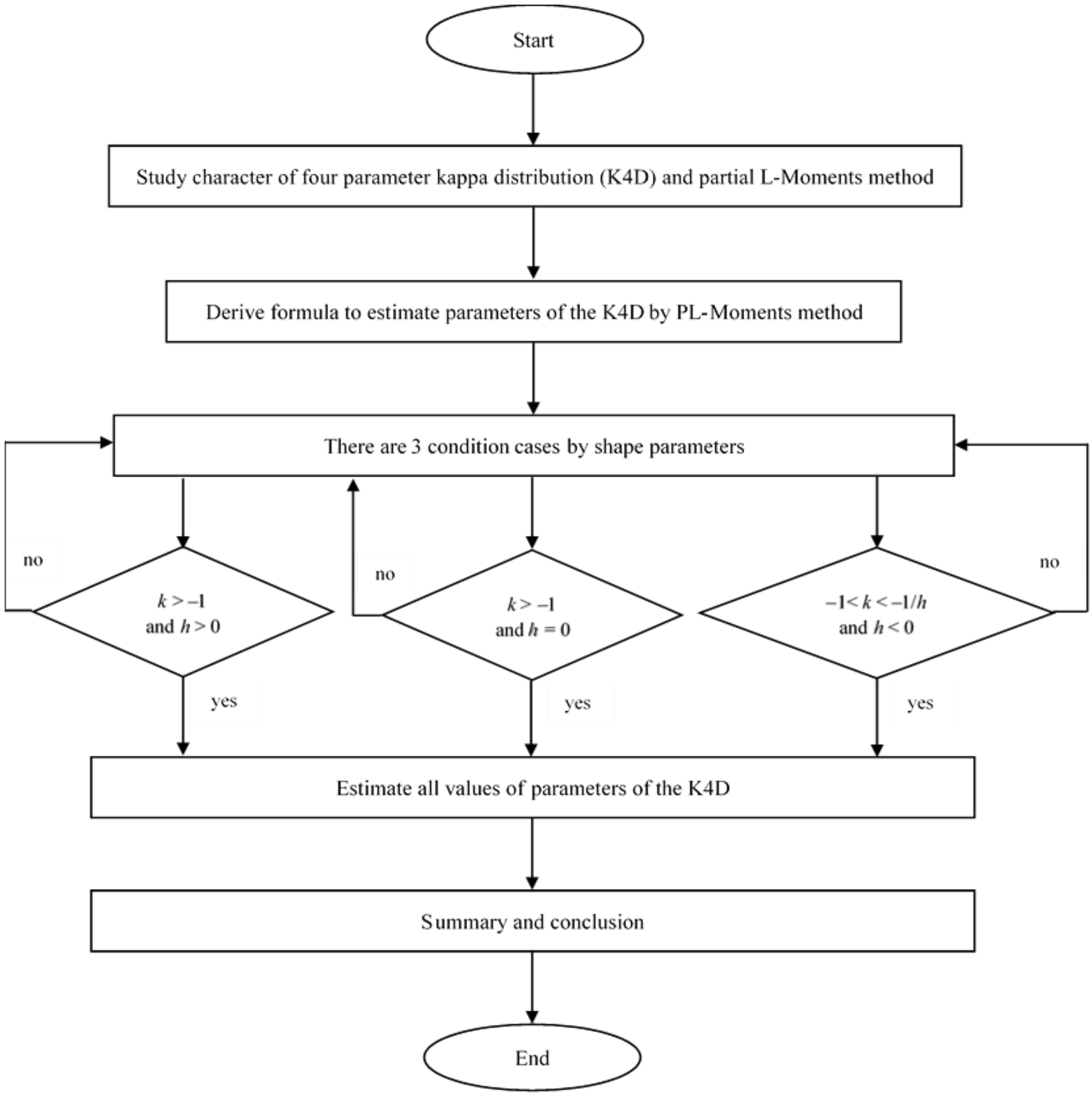

Figure 2. The procedures of the research methodology

From above 6 cases, the cdf and qf of the K4D are defined as (1) and (3), respectively. The PPWMs ($\beta_r^{/}$) of the K4D are derived as:

when $k \neq 0$

1. Case: $k > -1$ and $h > 0$;

$$\int_{F_0}^{1} x(F)F^r dF = \int_{F_0}^{1}\left\{\xi + \frac{\alpha}{k}\left[1-\left(\frac{1-F^h}{h}\right)^k\right]\right\}F^r dF = \frac{1}{r+1}\left(\xi+\frac{\alpha}{k}\right)(1-F_0^{r+1}) - \frac{\alpha}{k}\int_{F_0}^{1}\left(\frac{1-F^h}{h}\right)^k F^r dF \tag{13}$$

consider the second term of Equation 13,

let $v = 1 - F^h$, then

$$\frac{\alpha}{k}\int_{F_0}^{1}\left(\frac{1-F^h}{h}\right)^k F^r dF = \frac{1}{h^{k+1}}\frac{\alpha}{k}\int_0^{1-F_0^h} v^k (1-v)^{\frac{r+1}{h}-1} dv = \frac{1}{h^{k+1}}\frac{\alpha}{k} B_{1-F_0^h}\left(k+1, \frac{r+1}{h}\right). \tag{14}$$

Substitute Equation 14 into Equation 13, then

$\int_{F_0}^{1} x(F)F^r dF = \frac{1}{r+1}\left(\xi+\frac{\alpha}{k}\right)(1-F_0^{r+1}) - \frac{1}{h^{k+1}}\frac{\alpha}{k} B_{1-F_0^h}\left(k+1, \frac{r+1}{h}\right)$. So,

$$\beta_r^{/} = \frac{1}{r+1}\left(\xi+\frac{\alpha}{k}\right) - \frac{1}{(1-F_0^{r+1})}\frac{\alpha}{kh^{k+1}}B_{1-F_0^h}\left(k+1,\frac{r+1}{h}\right) \tag{15}$$

2. Case: $k > -1$ and $h = 0$;

$$\int_{F_0}^{1} x(F)F^r dF = \int_{F_0}^{1}\left\{\xi+\frac{\alpha}{k}[1-(-\ln F)^k]\right\}F^r dF = \frac{1}{r+1}\left(\xi+\frac{\alpha}{k}\right)(1-F_0^{r+1}) - \frac{\alpha}{k}\int_{F_0}^{1}(-\ln F)^k F^r dF \tag{16}$$

consider the second term of Equation 16,

let $u = -\ln F$, then

$\frac{\alpha}{k}\int_{F_0}^{1}(-\ln F)^k F^r dF = \frac{\alpha}{k}\int_{0}^{-\ln F_0} u^k e^{-u(r+1)}du$

and let $v = -(r+1)u$, then

$$\frac{\alpha}{k}\int_{0}^{-\ln F_0} u^k e^{-u(r+1)}du = \frac{1}{(r+1)^{k+1}}\frac{\alpha}{k}\int_{0}^{-(r+1)\ln F_0} v^{(k+1)-1}e^{-v}dv = \frac{1}{(r+1)^{k+1}}\frac{\alpha}{k}[\gamma(k+1,-(r+1)\ln F_0)]. \tag{17}$$

Substitute Equation 17 into Equation 16, then

$\int_{F_0}^{1} x(F)F^r dF = \frac{1}{r+1}\left(\xi+\frac{\alpha}{k}\right)(1-F_0^{r+1}) - \frac{1}{(r+1)^{k+1}}\frac{\alpha}{k}[\gamma(k+1,\ (r+1)\ln F_0)]$. So,

$$\beta_r^{/} = \frac{1}{r+1}\left(\xi+\frac{\alpha}{k}\right) - \frac{1}{(r+1)^{k+1}(1-F_0^{r+1})}\frac{\alpha}{k}[\gamma(k+1,-(r+1)\ln F_0)] \tag{18}$$

3. Case: $-1 < k < -\frac{1}{h}$ and $h < 0$;

$$\int_{F_0}^{1} x(F)F^r dF = \int_{F_0}^{1}\left\{\xi+\frac{\alpha}{k}\left[1-\left(\frac{1-F^h}{h}\right)^k\right]\right\}F^r dF = \frac{1}{r+1}\left(\xi+\frac{\alpha}{k}\right)(1-F_0^{r+1}) - \frac{\alpha}{k}\int_{F_0}^{1}\left(\frac{F^h-1}{-h}\right)^k F^r dF \tag{19}$$

consider the second term of Equation 19,

$\frac{\alpha}{k}\int_{F_0}^{1}\left(\frac{F^h-1}{-h}\right)^k F^r dF = \frac{\alpha}{k}\left(\frac{1}{-h}\right)^k\int_{F_0}^{1}(F^h-1)^k F^r dF$

let $u = F^h - 1$, then

$$\frac{\alpha}{k}\left(\frac{1}{-h}\right)^k\int_{F_0}^{1}(F^h-1)^k F^r dF = \frac{\alpha}{k}\left(\frac{1}{-h}\right)^{k+1}\int_{0}^{F_0^h-1} u^k(u+1)^{\frac{r+1}{h}-1}du = \frac{1}{(-h)^{k+1}}\frac{\alpha}{k}B_{F_0^h-1}\left(k+1,-\frac{r+1}{h}-k\right). \tag{20}$$

Substitute Equation 20 into Equation 19, then

$\int_{F_0}^{1} x(F)F^r dF = \frac{1}{r+1}\left(\xi+\frac{\alpha}{k}\right)(1-F_0^{r+1}) - \frac{\alpha}{k(-h)^{k+1}}B_{F_0^h-1}\left(k+1,\ -\frac{r+1}{h}-k\right)$. So,

$$\beta_r^{/} = \frac{1}{r+1}\left(\xi+\frac{\alpha}{k}\right) - \frac{1}{(1-F_0^{r+1})}\frac{\alpha}{k(-h)^{k+1}}B_{F_0^h-1}\left(k+1,-\frac{r+1}{h}-k\right). \tag{21}$$

Therefore, in case of $k \neq 0$ we can conclude as follows;

$$(r+1)\beta_r^{/} = \begin{cases} \xi+\frac{\alpha}{k}\left[1-\frac{r+1}{(1-F_0^{r+1})}\frac{1}{h^{k+1}}B_{1-F_0^h}\left(k+1,\frac{r+1}{h}\right)\right]; k>-1, h>0 \\ \xi+\frac{\alpha}{k}\left[1-\frac{1}{(r+1)^k(1-F_0^{r+1})}[\gamma(k+1,-(r+1)\ln F_0)]\right]; k>-1, h=0 \\ \xi+\frac{\alpha}{k}\left[1-\frac{r+1}{(1-F_0^{r+1})}\frac{1}{(-h)^{k+1}}B_{F_0^h-1}\left(k+1,-\frac{r+1}{h}-k\right)\right]; -1<k<-\frac{1}{h}, h<0 \end{cases}$$

In a similar way, when $k = 0$ we get

$$(r+1)\beta_r^{/} = \begin{cases} \xi+\alpha\left[\ln h+\frac{1}{1-F_0^{r+1}}D(a)\Big|_0^{1-F_0^h}\right]; h>0 \\ \xi+\frac{\alpha}{(1-F_0^{r+1})}[\varepsilon+\ln(r+1)+\ln(-\ln F_0)F_0^{r+1}+E_1[-(r+1)\ln F_0]]; h=0. \\ \xi+\alpha\left[\ln(-h)+\frac{1}{1-F_0^{r+1}}H(b)\Big|_0^{F_0^h-1}\right]; h<0 \end{cases}$$

where 1. $D(a)=(1-a)^{\frac{r+1}{h}}\ln a+\frac{h(1-a)^{\frac{r+1}{h}+1}}{r+1+h}\,{}_2F_1(1,\frac{r+1}{h}+1;\frac{r+1}{h}+2;1-a)$.

2. ${}_2F_1(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1\frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a}dt$ is a hypergeometric function.

3. $E_1[Z]=\int_z^\infty\frac{e^{-\theta}}{\theta}d\theta=-\varepsilon-\ln z-\sum_{n=1}^{\infty}\frac{(-1)^n z^n}{nn!};\ |\arg z|<\pi$ when $\varepsilon=0.5772$ is Euler's constant.

4. $H(b)=(b+1)^{\frac{r+1}{h}}\ln b+\frac{h(b+1)^{\frac{r+1}{h}+1}}{r+1+h}\,{}_2F_1(1,\frac{r+1}{h}+1;\frac{r+1}{h}+2;b+1)$.

By the formula of $(r+1)\beta_r^{/}$, we can compute the first four PL-Moments ($\lambda_1^{/},\lambda_2^{/},\lambda_3^{/}$ and $\lambda_4^{/}$), PL-skewness and PL-kurtosis from Equations 6 to 12, respectively.

4-2- PL-Moments Parameter Estimation for the K4D

Under the assumption that F_0 is known, Wang [9] presented formula of the unbiased estimator for $\beta_r^{/}$ as;$r=0,1,2,\ldots$

$$\hat{\beta}_r^{/}=\frac{1}{1-F_0^{r+1}}\frac{1}{n}\sum_{i=1}^{n}\frac{(i-1)(i-2)\ldots(i-r)}{(n-1)(n-2)\ldots(n-r)}x_{(i)}^{/} \tag{22}$$

where $x_{(i)}^{/}=\begin{cases}0;\,x_{(i)}\le x_0\\ x_{(i)};\,x_{(i)}>x_0\end{cases}$ and $x_{(1)}\le x_{(2)}\le\ldots\le x_{(n)}$

If F_0 is unknown then $\hat{F}_0=\frac{n_0}{n}$, when n_0 is the number of the sample which do not exceed the threshold x_0 of the n event.

Therefore, in this study, we derived the first four sample PL-Moments, $\hat{\lambda}_1^{/},\hat{\lambda}_2^{/},\hat{\lambda}_3^{/}$and $\hat{\lambda}_4^{/}$, are unbiased estimators of PL-Moments, $\lambda_1^{/},\lambda_2^{/},\lambda_3^{/}$and $\lambda_4^{/}$ as follows:

$$\hat{\lambda}_1^{/}=\frac{1}{n^2(1-\hat{F}_0)}\sum_{i=1}^{n}ix_{(i)}^{/}, \tag{23}$$

$$\hat{\lambda}_2^{/}=\frac{1}{n}\left[\frac{2}{(n-1)(1-\hat{F}_0^2)}\sum_{i=1}^{n-1}ix_{(i+1)}^{/}-\frac{1}{n(1-\hat{F}_0)}\sum_{i=1}^{n}ix_{(i)}^{/}\right], \tag{24}$$

$$\hat{\lambda}_3^{/}=\frac{1}{n}\left[\frac{6}{(n-1)(n-2)(1-\hat{F}_0^3)}\sum_{i=1}^{n-2}i(i+1)x_{(i+2)}^{/}-\frac{6}{(n-1)(1-\hat{F}_0^2)}\sum_{i=1}^{n-1}ix_{(i+1)}^{/}+\frac{1}{n(1-\hat{F}_0)}\sum_{i=1}^{n}ix_{(i)}^{/}\right], \tag{25}$$

$$\hat{\lambda}_4^{/}=\frac{1}{n}\left[\begin{array}{l}\frac{20}{(n-1)(n-2)(n-3)(1-\hat{F}_0^4)}\sum_{i=1}^{n-3}i(i+1)(i+2)x_{(i+3)}^{/}-\frac{30}{(n-1)(n-2)(1-\hat{F}_0^3)}\sum_{i=1}^{n-2}i(i+1)x_{(i+2)}^{/}\\+\frac{12}{(n-1)(1-\hat{F}_0^2)}\sum_{i=1}^{n-1}ix_{(i+1)}^{/}-\frac{1}{n(1-\hat{F}_0)}\sum_{i=1}^{n}ix_{(i)}^{/}\end{array}\right], \tag{26}$$

while the sample PL-Moments ratio, $\hat{\tau}_r^{/}$, are unbiased estimators of the PL-Moments ratio, $\tau_r^{/}$ as follows:

the sample PL-skewness, $\hat{\tau}_3^{/}=\frac{\hat{\lambda}_3^{/}}{\hat{\lambda}_2^{/}}$ and sample PL-kurtosis, $\hat{\tau}_4^{/}=\frac{\hat{\lambda}_4^{/}}{\hat{\lambda}_2^{/}}$. (27)

Hosking [16] has provided the conditions on the parameters of K4D for the existence of L-Moments (conditions 1 and 2) and for the uniqueness of the parameters (conditions 3 and 2). The PL-Moments of K4D are determined by the following four conditions below:

1. $k>-1$
2. if $h<0$, then $kh>-1$
3. $h>-1$
4. $k+0.752h>-1$

We used the above nonlinear constraints and augmented Lagrangian adaptive barrier minimization algorithm, a non-linear optimization algorithm, to estimate parameters ($\hat{\alpha}$, $\hat{k}$, and $\hat{h}$), which set the nonlinear objective function converting to zero, written as [24]:

$$f(\alpha,k,h)=(\lambda_2^{/}-\hat{\lambda}_2^{/})^2+(\lambda_3^{/}-\hat{\lambda}_3^{/})^2+(\lambda_4^{/}-\hat{\lambda}_4^{/})^2\ge 0 \tag{28}$$

Therefore, the estimates of $\hat{\xi}$ are obtained by plugging $\hat{\alpha}$, $\hat{k}$ and $\hat{h}$, from (28), into $\lambda_1^{/}$, we have shown derived formulas of $\hat{\xi}$ in 3 cases, when $k \neq 0$, as follows:

case $k > -1$ and $h > 0$,

$$\hat{\xi} = \hat{\lambda}_1^{/} - \frac{\hat{\alpha}}{\hat{k}}\left[1 - \frac{1}{\hat{h}^{\hat{k}+1}}\frac{1}{(1-\hat{F}_0)}B_{1-\hat{F}_0^{\hat{h}}}\left(\hat{k}+1,\frac{1}{\hat{h}}\right)\right], \tag{29}$$

case $k > -1$ and $h = 0$,

$$\hat{\xi} = \hat{\lambda}_1^{/} - \frac{\hat{\alpha}}{\hat{k}}\left[1 - \frac{1}{(1-\hat{F}_0)}\left[\gamma\left(\hat{k}+1,-ln\hat{F}_0\right)\right]\right], \tag{30}$$

case $-1 < k < -\frac{1}{h}$ and $h < 0$,

$$\hat{\xi} = \hat{\lambda}_1^{/} - \frac{\hat{\alpha}}{\hat{k}}\left[1 - \frac{1}{(\hat{h})^{\hat{k}+1}}\frac{1}{(1-\hat{F}_0)}B_{\hat{F}_0^{\hat{h}}-1}\left(\hat{k}+1,-\frac{1}{\hat{h}}-\hat{k}\right)\right]. \tag{31}$$

From the results, we got the new formula to estimate the parameters of K4D using the PL-Moments approach.

5- Conclusions

The four-parameter Kappa distribution, K4D, is a well-known and flexible applicable distribution to the real data including extreme events or skewed events. Form the literature review, we found that the PL-Moments method gives better results of parameter estimation than L-Moments of the generalized extreme value distribution, GEV, which is a three parameter Kappa distribution when $k \neq 0$and $h = 0$. Therefore, in this study, the new modification of formulas to estimate parameters based on the PL-Moments for K4D has been proposed. We have derived the Partial Probability-Weighted Moments (PPWMs) of the K4D ($\beta_r^{/}$): there are 6 cases as follows:

$$\text{when } k \neq 0,\ (r+1)\beta_r^{/} = \begin{cases} \xi + \frac{\alpha}{k}\left[1 - \frac{r+1}{(1-F_0^{r+1})}\frac{1}{h^{k+1}}B_{1-F_0^{h}}\left(k+1,\frac{r+1}{h}\right)\right]; k > -1, h > 0 \\ \xi + \frac{\alpha}{k}\left[1 - \frac{1}{(r+1)^k(1-F_0^{r+1})}\left[\gamma(k+1,-(r+1)\ln F_0)\right]\right]; k > -1, h = 0 \\ \xi + \frac{\alpha}{k}\left[1 - \frac{r+1}{(1-F_0^{r+1})}\frac{1}{(-h)^{k+1}}B_{F_0^{h}-1}\left(k+1,-\frac{r+1}{h}-k\right)\right]; -1 < k < -\frac{1}{h}, h < 0 \end{cases},$$

$$\text{when } k = 0,\ (r+1)\beta_r^{/} = \begin{cases} \xi + \alpha\left[\ln h + \frac{1}{1-F_0^{r+1}}D(a)\Big|_0^{1-F_0^{h}}\right]; h > 0 \\ \xi + \frac{\alpha}{(1-F_0^{r+1})}\left[\varepsilon + ln(r+1) + ln(-\ln F_0)F_0^{r+1} + E_1[-(r+1)\ln F_0]\right]; h = 0. \\ \xi + \alpha\left[ln(-h) + \frac{1}{1-F_0^{r+1}}H(b)\Big|_0^{F_0^{h}-1}\right]; h < 0 \end{cases}$$

We have derived new formulas to estimate parameters of K4D with the PL-Moments method ($\hat{\xi}$, $\hat{\alpha}$, $\hat{k}$, and $\hat{h}$), when $\hat{\xi}$ is an unbiased estimator of a location parameter, $\hat{\alpha}$ is an unbiased estimator of a scale parameter, and $\hat{k}$ and $\hat{h}$ are unbiased estimators of shape parameters. There are 3 cases when $k \neq 0$ to derive the estimation of the shape parameters (k, h), case 1, $k > -1$ and $h > 0$, case 2, $k > -1$ and $h = 0$, and case 3,$-1 < k < -\frac{1}{h}$ and $h < 0$.

We used the nonlinear constraints and augmented Lagrangian adaptive barrier minimization algorithm to estimate the parameters ($\hat{\alpha}$, $\hat{k}$, and$\hat{h}$), followed by condition of $f(\alpha,k,h) = (\lambda_2^{/} - \hat{\lambda}_2^{/})^2 + (\lambda_3^{/} - \hat{\lambda}_3^{/})^2 + (\lambda_4^{/} - \hat{\lambda}_4^{/})^2 \geq 0$.

Therefore, the estimates of $\hat{\xi}$ are obtained by plugging $\hat{\alpha}$, $\hat{k}$ and $\hat{h}$, from Equation 28, into $\lambda_1^{/}$, we have shown derived formulas of $\hat{\xi}$ in 3 cases, when $k \neq 0$, as follows:

case $k > -1$ and $h > 0$, $\hat{\xi} = \hat{\lambda}_1^{/} - \frac{\hat{\alpha}}{\hat{k}}\left[1 - \frac{1}{\hat{h}^{\hat{k}+1}}\frac{1}{(1-\hat{F}_0)}B_{1-\hat{F}_0^{\hat{h}}}\left(\hat{k}+1,\frac{1}{\hat{h}}\right)\right]$,

case $k > -1$ and $h = 0$, $\hat{\xi} = \hat{\lambda}_1^{/} - \frac{\hat{\alpha}}{\hat{k}}\left[1 - \frac{1}{(1-\hat{F}_0)}\left[\gamma\left(\hat{k}+1,-ln\hat{F}_0\right)\right]\right]$,

case $-1 < k < -\frac{1}{h}$ and $h < 0$, $\hat{\xi} = \hat{\lambda}_1^{/} - \frac{\hat{\alpha}}{\hat{k}}\left[1 - \frac{1}{(-\hat{h})^{\hat{k}+1}}\frac{1}{(1-\hat{F}_0)}B_{\hat{F}_0^{\hat{h}}-1}\left(\hat{k}+1,-\frac{1}{\hat{h}}-\hat{k}\right)\right]$.

Moreover, this new derived formula can be applied to predict return levels for the planning and management of many extreme events. For our future research, we will apply this parameter estimation formula to real events, for example, extreme temperature data, flood data, wave data, and wind data.

6- Declarations

6-1- Author Contributions

Conceptualization, P.G. and P.B.; methodology, P.G.; validation, P.G. and M.C.; formal analysis, P.G.; investigation, P.G. and M.C.; resources, P.G. and M.C.; writing—original draft preparation, P.G. and M.C.; writing—review and editing, M.C.; corresponding author, M.C.; visualization, P.G.; project administration, P.G.; funding acquisition, P.G. All authors have read and agreed to the published version of the manuscript.

6-2- Data Availability Statement

The data presented in this study are available in the article.

6-3- Funding

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6-4- Institutional Review Board Statement

Not applicable.

6-5- Informed Consent Statement

Not applicable.

6-6- Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this manuscript. In addition, the ethical issues, including plagiarism, informed consent, misconduct, data fabrication and/or falsification, double publication and/or submission, and redundancies have been completely observed by the authors.

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