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Analysis of Four-Species Diffusive and Non-Diffusive Food Chains Using Artificial Neural Networking

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Abstract

This study uncovers the findings of a four-species food chain model, focusing on its equilibrium points, global stability, and population dynamics. Through rigorous mathematical analysis, we identify the equilibrium points of the model and investigate the global stability of the coexistence equilibrium point. We present the existence conditions for all equilibrium points and assess the stability characteristics of the coexistence fixed point. Time series solutions offer a captivating perspective on the dynamic behavior of a system. Our investigation into the effects of parameters provides the fluctuations in population density, with specific parameters exerting significant influence as a result of the random movement of linked species. Understanding the need for taking account of diffusion-dominated situations, the diffusive version of the model is developed and analyzed. By constructing a numerical system with three-time levels (*n-1*, *n*, and *n+1*), its stability can potentially be tested thoroughly using the Von Neumann stability criterion. Numerical simulations and graphs depict the system's dynamic interaction. We also examine how diffusion coefficients affect population density, creating remarkable charts that show interactive species relationships. We also identify exciting bifurcation occurrences in the system, which helps us comprehend its complex dynamics. Predator-prey systems can be studied using Artificial Neural Networks (ANNs) to handle complexity, discover patterns, and predict future dynamics. ANNs can predict population dynamics and assess various parameters by analyzing prior data. Their adaptability lets them improve forecasts over time, improving management methods and ecosystem balance. We use ANN methods to see how specific parameters affect interacting species population dynamics.

Keywords:

Food Chain;

Lyapunov Function;

Global Stability;

Bifurcation;

Explicit Numerical Scheme;

Artificial Neural Network.

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1- Introduction

When both organisms interact and do not damage each other, both gain from it in terms of food, shelter, and growth. For example, we say that the relationship is mutualistic. Such interactions have gained much attention from contemporary society during the past several years [1-3], contrary to other significant ecological interactions that cast negative influences, such as predation and competition. Cleaner fish, pollination, seed dispersal mechanisms, gut flora, and nitrogen fixation are just a few instances of a mutualistic association that are mentioned in the traditional ecological theory of natural selection and habitat separation, which also accounts for metapopulations. Positive interactions can be spotted naturally due to interaction with the third population in the competitive or predator-prey relationship [4, 5]. Before: It is revealed that predators are of utmost importance in mutualism occurring in predator-prey relations and described by formulating mathematical modeling [6-8].

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The populations mutualistic to predators could survive without mutualists. This means mutualism exists in a population irrespective of isolated populations, which are difficult to witness because of the continual migration process. The authors investigated the mutualism that occurs in prey with the two-dimensional mutualism mode [9-11]; however, a three-dimensional model was formulated by Rai et al. [12], in which mutualism occurred due to third-personality interaction. The three-dimensional mutualistic model in which mutualism occurred in prey [7, 13, 14]. Following Thirthar et al. [8], these models were further strengthened through their integration with the food chain.

Qualitative traits of ecological systems made them challenging to understand with respect to mutualism, which is very diverse, obligatory and needs:

- The mechanism by which one species benefits the other.
- The number of species interacting to attain mutualism between them.

The complexity of ecological problems that organisms face might vary depending upon the variation that mutualistic interaction causes. Several factors are predominant in mutualism: shelter, organic nutrients, dispersal of gametes, competition, and predation, and imparting mutualistic benefits to the population [4]. Qualitative mutualistic systems are of different kinds and benefit involved species differently.

Mutualists of prey may reduce the predation of their predators or compete with them. A mutualist of a predator may increase predation on the prey or stimulate the prey to more rapid growth. Mutualists can assist a species in out-competing its predators by adding it directly, competing with competitors, or preying on predators. Because of these, it is essential to analyze multispecies models. Mutualistic benefits involve the direct modification of abiotic components and the circulation of nutrients among governing bodies. It has been studied that mutualistic interaction might have consisted of three or more species [15]. A mutualist prey competes with its predator and decreases the predation of its predator, whereas a mutualistic predator causes vigorous growth of its prey. A mutualist species competes with its predators directly [16].

Taking account of the complexity and diversity of a mutualistic system, it isn't easy to get peculiar results after applying a unified approach. Initially, mutualism was discussed in terms of the two-species model. Later, multispecies models were formulated in which the number of species extended to the value n reasons for considering the multispecies model:

- The qualitative behaviour of systems can never be explained through a single-species mutualistic model.
- Models with two-species mutualistic relations need additional density-dependent mutualism coefficients, even if the system defines the model's complexity.
- Appropriate forms of the function appear when such models are analyzed.
- The multispecies model provides field ecologists with a new platform to reveal variable aspects of research.

The ecosystem combines several species, and a single-species ecosystem is rare. However, during the past few centuries, ecologists focused on two- or three-species systems, such as predator-prey and food chain systems [17, 18]. Plenty of naturally occurring phenomena are beyond the two- or three-species systems but consist of many interacting species. It is of utmost importance to formulate theoretical methods to study such complex systems [19]. Owaidy et al. [20] explored the presence of bounded solutions and the stability of equilibrium points in a four-level generalized food-chain model.

Forest and wildlife management authorities established policies and harvesting laws in developed countries to prevent ecological niches. The fundamental aspect of these preventions is to protect the integrity of nature and the environment without economic default. Therefore, it is essential to develop theoretical methods for optimal harvesting techniques to conserve the environment. In this respect, Tuerxun et al. [21] investigated a stochastic two-predator and one-prey system with distributed delays, harvesting, and Levy jumps. The main objectives were to analyze optimal harvesting strategy, and environmental changes mainly influenced the results.

The ecosystem is a self-regulatory system of biotic and abiotic components to attain functional stability. Communities comprising one or two species are very rare in nature due to the process of migration, which led to the coexistence of many species to stabilize the natural ecosystem [17, 22]. Scientists have worked over the past few years to develop models of two or three species behaving mutualistically, while multispecies models also exist [19, 23]. It is revealed in Hastings & Powell [24] that the coexistence of species is not only studied in terms of singular stability and orbits but also in terms of quasi-periodic or strange attractors.

Several simulation studies depicted that food chain models exhibit chaotic dynamics due to periodic doubling [25-31]. In Hastings & Powell [24], the authors demonstrated the presence of a tea-cup strange attractor in three trophic levels of a food chain. The biological feasibility of strange attractors utilized by several authors in their studies has been questioned in McCann & Yodzis [27]. In contrast, parameters used by Scheffer [30] for plankton and Wilder et al. [31] to study gypsy moths are biologically feasible. Facts about strange attractors were combined and incorporated into analyses of the food chain and web (see [25, 27, 28]). The above studies are strong witnesses to deeper food chain/web complexity.

In Gakkhar & Naji [32], the authors modified the given model by altering the functional responses through numerical simulation and bringing it to the level where it becomes biologically feasible. The presence of chaos within the nature of species and that chaotic condition in the food chain and food web are discussed in Gakkhar & Naji [33, 34]. In El-Owaidy & Ammar [20], the major objectives followed by the author were the analysis of the existence of a bounded solution and the measurement of the stability of equilibrium points, which were not enough to combat the current environmental challenges. The study by Pal et al. [35] explores a three-species food chain model comprising prey, an intermediate predator, and a top predator, emphasizing the effects of fear responses and foraging constraints on their interactions. Fear triggered by the intermediate predator impacts the prey's reproduction rate and intra-specific competition, while the presence of the top predator hinders the foraging efficiency of the intermediate predator. Through equilibrium analysis and stability investigations, the study demonstrates how behavioral and ecological factors influence population dynamics and food web resilience. In the study, Saikumar et al. (2024) [36] investigate the impact of microplastics on estuarine food chains through trophic transfer. They analyze how microplastics accumulate and move through various trophic levels, leading to bioaccumulation and potential ecological risks. The study highlights the need for strategies to mitigate the effects of microplastics in estuarine environments.

Gao et al. [37] used the Eco tracer module from the EcoPath with the EcoSim (EwE) model to study microplastics (MPs) in the marine food web of Haizhou Bay, Jiangsu Province, China, over 20 years. They linked environmental plastic inflow with MPs in organisms to simulate their distribution and found that top consumers accumulated more MPs, while primary consumers showed decreased MP concentrations. Functional groups demonstrated trophic magnification with no bio-dilution. Reducing plastic inflows could effectively lessen MP pollution in coastal waters. Gomes et al. [38] analyzed the impacts of marine heat waves on the Northeast Pacific Ocean using time series abundance data, functional groups, and diet information. They built two food web models using the Ecotran extension of Ecopath, comparing pre- and post-heat wave conditions. The study revealed significant changes in trophic relationships, especially among gelatinous taxa like pyrosomes, and potential risks for threatened and harvested species due to altered ecosystem structure and function. The author in Xu et al. [39] emphasizes the interactions between microplastics (MPs) and organic pollutants (OPs), focusing on their potential risks through bioaccumulation and biomagnification in the food chain. There is limited data on MPs/OPs in food pollution, highlighting the need for further research on their impact on human health and food quality. The role of factors like temperature and pH in influencing pollutant behavior, particularly how MPs affect OP adsorption, is also addressed. Most of the food chains in the literature focus on three species of food chains; the food chains having four species are rarely discussed. The present study focuses on four species, incorporating diffusive effects. Additionally, the ANN has been incorporated to analyze the system.

The paper is organized as follows: Section 2 introduces the formulation of the model using ordinary differential equations (ODEs) and provides details about the parameters. Section 3 explores the equilibrium points of the model, outlining the conditions for their existence and conducting a stability analysis. Section 4 focuses on the integration of Artificial Neural Networks (ANN) into the model framework. Section 5 extends the model to include its diffusive version, providing a comprehensive analysis of dynamics under diffusion. Section 6 develops a numerical scheme and performs a Von Neumann stability analysis. Finally, Section 7 concludes the study by summarizing key findings and suggesting directions for future research.

Figures 1 and 2, respectively, represent the flow charts of the mathematical model and the methodology of the study.

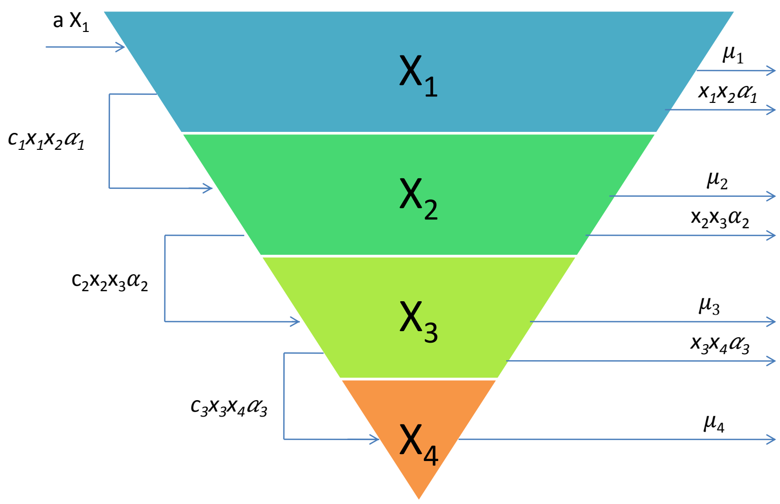


Figure 1. **Visualizing the Mathematical Food Chain Model**

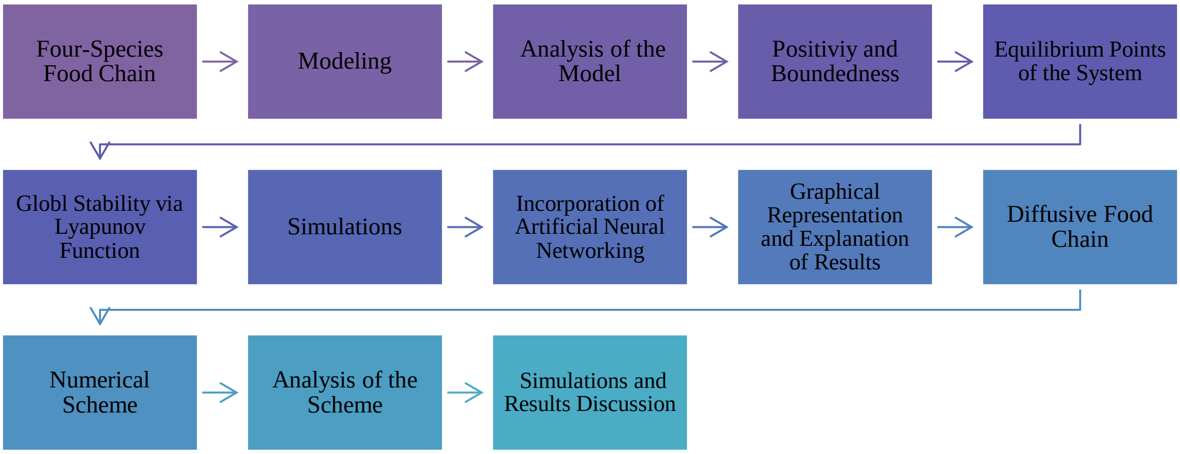


Figure 2. Flow Chart of the Methodology

2- The Food Chain Mathematical Model

The food chains and webs have different trophic levels. Generally, the food chains are represented by different trophic levels. These levels may include basal prey, primary, secondary, and tertiary consumers. We formulate a mathematical model to observe the relationship among species in a habitat. Four species are considered to model the food chain. We assume that the basal prey grows at a growth rate $'a'$. It is assumed that the predation rate by the species involved is α_i $(i = 1, 2, 3, 4)$. The conversion rate of species to the next species under the effects of predation is c_i $(i = 1, 2, 3,)$. It is assumed that the natural mortality of the species is μ_i $(i = 1, 2, 3, 4)$.

The parameters specifying growth rate, death rate conversion rate, etc., used in the model formulation are assumed to be non-negative. The system of ODEs developed under these assumptions is as follows.

$$\frac{dx_1(t)}{dt} = x_1(t)(a - x_2(t)\alpha_1 - \mu_1), t > 0, \tag{1}$$

$$\frac{dx_2(t)}{dt} = x_2(t)(c_1 x_1(\mathrm{t})\alpha_1 - x_3(t)\alpha_2 - \mu_2), t > 0 \tag{2}$$

$$\frac{dx_3(t)}{dt} = x_3(t)(c_2 x_2(t)\alpha_2 - x_4(t)\alpha_3 - \mu_3), t > 0, \tag{3}$$

$$\frac{dx_4(t)}{dt} = x_4(t)(c_3 x_3(t)\alpha_3 - \mu_4), t > 0 \tag{4}$$

where, $x_i(0) \geq 0, i = 1, 2, 3, 4$.

Table 1 shows the parameters, and their description used in the formulation of the system.

Table 1. Parameters and their physical meanings

Parameters	Physical meaning
x_1	Density of basal prey
x_2	Density of mid-predator-1
x_3	Density of mid-predator-2
x_4	Density of top predator
a	Growth rate of basal prey
α_1	Death rate of basal prey due to predation
α_2	Death rate of mid-predator-1 due to predation
α_3	Death rate of mid-predator-2
c_1	Biomass conversion parameter for basal prey to predator-1
c_2	Biomass conversion parameter for medium predator-1 to predator-2
c_3	Biomass conversion parameter for medium predator-2 to top predator
μ_1	Natural death rate of basal prey
μ_2	Natural death rate of medium predator-1
μ_3	Natural death rate of medium predator-2
μ_4	Natural death rate of top predator

3- Analysis of the Model

In this section, we investigate the positivity, boundedness, and stability aspects of the equilibrium points within the framework of the model (1-4). We discover how equilibrium points are maintained within the system and investigate their behavior concerning positivity and boundedness. Additionally, we scrutinize the stability of these fixed points, assessing how the system responds to perturbations and whether equilibrium is maintained over time. This examination highlights the long-term behavior, dynamics, and resilience of the system.

3-1- Positivity and Boundedness

In order to determine particular attributes, population models display certain essential features. When trying to make sense of population dynamics and behaviors in ecological systems, these traits are crucial. We may learn about the population models' development patterns, stability, and reactions to environmental changes by investigating these key characteristics.

Theorem 1: All solutions of the systems (1-4) that start in R_+^4 will always remain positive.

Proof: The first equation of system (1-4) gives the following result.

$$\frac{dx_1(t)}{dt} = x_1(t)(a - x_2(t)\alpha_1 - \mu_1)$$

$$x_1(t) = x_1(0)exp\left[\int_0^t (a - x_2(\theta)\alpha_1 - \mu_1)d\theta\right]$$

$$\Rightarrow x_1(t) \geq 0$$

Similarly, Equation 2 results in the following equation.

$$x_2(t) = x_2(0)exp\left[\int_0^t \{(c_1x_1(\theta)\alpha_1 - x_3(\theta)\alpha_2 - \mu_2)\}d\theta\right]$$

$$\Rightarrow x_2(t) \geq 0$$

Now, Equation 3 gives us the following result.

$$x_3(t) = x_3(0)exp\left[\int_0^t (c_2x_2(\theta)\alpha_2 - x_4(\theta)\alpha_3 - \mu_3)d\theta\right]$$

$$\Rightarrow x_3(t) \geq 0$$

$$x_4(t) = x_4(0)exp\left[\int_0^t (c_3x_3(\theta)\alpha_3 - \mu_4)d\theta\right]$$

$$\Rightarrow x_4(t) \geq 0$$

Hence, the theorem is proved.

Theorem 2: All solutions of the systems (1-3) are bounded.

Proof: Consider system (1-4). The initial equation of the system gives the subsequent result.

$$\frac{dx_1(t)}{dt} = x_1(t)(a - x_2(t)\alpha_1 - \mu_1)$$

$$\frac{dx_1(t)}{dt} \leq ax_1(t)$$

$$x_1(t) \leq K_1\, e^{at}$$

where $K_1 = e^{c_1}$, with c_1 as constant of integration.

Similarly, the other equations provide the following outcomes.

$$\frac{dx_2(t)}{dt} \leq c_1\alpha_1x_1(t)x_2(t)$$

$$\frac{dx_2(t)}{x_2(t)} \leq c_1\alpha_1K_1\, e^{at}d(t)$$

$$x_2(t) \leq K_2\, e^{at}$$

where, $K_2 = e^{c_2}c_1\alpha_1K_1$ with c_2 as constant of integration.

$$\frac{dx_3(t)}{dt} \leq c_2\alpha_2x_2(t)x_3(t)$$

$$\frac{dx_3(t)}{x_3(t)} \leq c_2\alpha_2x_2(t)dt$$

$$\frac{dx_3(t)}{x_3(t)} \leq c_2\alpha_2K_2\, e^{at}$$

$$x_3(t) \leq K_3\, e^{at}$$

where, $K_3 = e^{c_3}c_2\alpha_2K_2$ with c_2 as constant of integration.

Considering Equation 4, we have the following result.

$\frac{dx_4(t)}{dt} \leq c_3\alpha_3 x_3(t) x_4(t)$

It is easy to see that,

$x_4(t) \leq K_4\, e^{at}$.

where, $K_4 = e^{c_4} c_3 \alpha_3 K_3$ with c_3 as constant of integration.

Hence, the theorem is proved

3-2- Stability Analysis

Here, we do an extensive mathematical examination of the model that has been provided in (1-4). The goal of this section is to locate the model's equilibrium points and elucidate the necessary conditions for their existence. By thoroughly examining the dynamics of the model, we aim to pinpoint the pivotal points at which the system achieves stability. This analytical approach provides the details of how the model behaves and maintains stability across different circumstances. Furthermore, we look at the implications of these equilibrium points. This precise mathematical scrutiny enriches our understanding of the dynamics of the model. It facilitates a deeper exploration of its broader implications within the context of the phenomenon under study.

3-3- Equilibrium Points of Model

We have the following system of equations for computing the equilibrium points of the proposed model (1-4).

$$0 = x_1(t)(a - x_2(t)\alpha_1 - \mu_1) \quad (5)$$

$$0 = x_2(t)(c_1 x_1(\mathrm{t})\alpha_1 - x_3(t)\alpha_2 - \mu_2) \quad (6)$$

$$0 = x_3(t)(c_2 x_2(t)\alpha_2 - x_4(t)\alpha_3 - \mu_3) \quad (7)$$

$$0 = x_4(t)(c_3 x_3(t)\alpha_3 - \mu_4) \quad (8)$$

After solving the system of Equation 5 to 8 simultaneously, we find that five equilibrium points are given as under.

a) Prey-free, top predator-free equilibrium point

$E_1 = \left(0, \frac{\mu_3}{c_2\alpha_2}, -\frac{\mu_2}{\alpha_2}, 0\right)$

This particular equilibrium point represents the extinction of basal prey and top predators. It is obvious that this point does not exist because $\mu_2 > 0$ and $\alpha_2 > 0$.

b) Prey-free, predator-1 free fixed point

$E_2 = \left(0, 0, \frac{\mu_4}{c_3\alpha_3}, -\frac{\mu_3}{\alpha_3}\right)$

This particular equilibrium point represents the extinction of basal prey and medium predator-1. It is obvious that this point does not exist because $\mu_3 > 0$ and $\alpha_3 > 0$.

c) Coexistence Fixed Point

$E_3 = \left(\frac{c_3\alpha_3\mu_2 + \alpha_2\mu_4}{c_1c_3\alpha_1\alpha_3}, \frac{a-\mu_1}{\alpha_1}, \frac{\mu_4}{c_3\alpha_3}, \frac{c_2\alpha_2(a-\mu_1)-\alpha_1\mu_3}{\alpha_1\alpha_3}\right)$

The equilibrium point guarantees the existence of all species under the following conditions.

$a > \mu_1,\ c_2\alpha_2(a - \mu_1) > \alpha_1\mu_3$.

d) Trivial Fixed Point

$E_4 = (0, 0, 0, 0)$

This equilibrium point shows the extinction of all the species from the habitat. From an ecological point of view, this point is not of interest as it leads to habitat destruction.

e) Predator-2, top predator-free fixed point

$E_5 = \left(\frac{\mu_2}{c_1\alpha_1}, \frac{a-\mu_1}{\alpha_1}, 0, 0\right)$

The equilibrium point guarantees the extinction of two predators, i.e., medium predator and top predator. This point exists if the growth rate of prey is greater than its natural death rate.

3-4- Global Stability

To deal with the global stability of coexistence fixed point we have the following theorem.

Theorem 3: For $x_i \geq x_i^*$, $(i = 1, 2, 3, 4)$, the equations (1-4) are globally asymptotically stable for the coexistence equilibrium point if the following hold.

$0 \leq c_i < 1, (i = 1, 2, 3)$

Proof: Consider the following Lyapunov function for the coexistence equilibrium point.

$$V(x_1, x_2, x_3, x_4) = k_1\left(x_1 - x_1^* - x_1^* \log\frac{x_1}{x_1^*}\right) + k_2\left(x_2 - x_2^* - x_2^* \log\frac{x_2}{x_2^*}\right) + k_3\left(x_3 - x_3^* - x_3^* \log\frac{x_3}{x_3^*}\right) + k_4\left(x_4 - x_4^* - x_4^* \log\frac{x_4}{x_4^*}\right) \quad (9)$$

where k_1, k_2, k_3 and k_4 are positive constants whose value is to be determined. The derivative of the above Equation 9 leads to the following result.

$$\frac{dV}{dt} = k_1\left(1 - \frac{x_1^*}{x_1}\right)\frac{dx_1}{dt} + k_2\left(1 - \frac{x_2^*}{x_2}\right)\frac{dx_2}{dt} + k_3\left(1 - \frac{x_3^*}{x_3}\right)\frac{dx_3}{dt} + k_4\left(1 - \frac{x_4^*}{x_4}\right)\frac{dx_4}{dt}. \quad (10)$$

We get the following form after some simplification.

$$\frac{dV}{dt} = k_1\left(\frac{x_1 - x_1^*}{x_1}\right)\frac{dx_1}{dt} + k_2\left(\frac{x_1 - x_1^*}{x_1}\right)\frac{dx_2}{dt} + k_3\left(\frac{x_1 - x_1^*}{x_1}\right)\frac{dx_3}{dt} + k_4\left(\frac{x_1 - x_1^*}{x_1}\right)\frac{dx_4}{dt} \quad (11)$$

By using Equations 1 to 5 in the above equation, we get the following form

$$\frac{dV}{dt} = k_1\left(\frac{x_1 - x_1^*}{x_1}\right)(x_1(t)(a - x_2(t)\alpha_1 - \mu_1)) + k_2\left(\frac{x_2 - x_2^*}{x_2}\right)(x_2(t)(c_1 x_1(t)\alpha_1 - x_3(t)\alpha_2 - \mu_2)) + k_3\left(\frac{x_3 - x_3^*}{x_3}\right)(x_3(t)(c_2 x_2(t)\alpha_2 - x_4(t)\alpha_3 - \mu_3)) + k_4\left(\frac{x_4 - x_4^*}{x_4}\right)\left(x_4(t)(c_3 x_3(t)\alpha_3 - \mu_4)\right). \quad (12)$$

$$\frac{dV}{dt} = k_1(x_1 - x_1^*)(a - x_2\alpha_1 - \mu_1) + k_2(x_2 - x_2^*)(c_1 x_1\alpha_1 - x_3\alpha_2 - \mu_2) + k_3(x_3 - x_3^*)(c_2 x_2\alpha_2 - x_4\alpha_3 - \mu_3) + k_4(x_4 - x_4^*)(c_3 x_3\alpha_3 - \mu_4) \quad (13)$$

By utilizing the fact that $\frac{dx_i^*}{dt} = 0$, for $(i = 1,2,3,4)$, we get the following.

$$\frac{dV}{dt} = k_1(x_1 - x_1^*)(x_2^*\alpha_1 - x_2\alpha_1) + k_2(x_2 - x_2^*)(c_1 x_1\alpha_1 - x_3\alpha_2 - c_1 x_1^*\alpha_1 + x_3^*\alpha_2) + k_3(x_3 - x_3^*)(c_2 x_2\alpha_2 - x_4\alpha_3 - c_2 x_2^*\alpha_2 + x_4^*\alpha_3) + k_4(x_4 - x_4^*) \quad (14)$$

Some simplification leads to the following result.

$$\frac{dV}{dt} = -k_1\alpha_1(x_1 - x_1^*)(x_2 - x_2^*) + k_2 c_1\alpha_1(x_2 - x_2^*)(x_1 - x_1^*) - k_2\alpha_2(x_2 - x_2^*)(x_3 - x_3^*) + k_3 c_2\alpha_2(x_3 - x_3^*)(x_2 - x_2^*) - k_3\alpha_3(x_3 - x_3^*)(x_4 - x_4^*) + k_4 c_3\alpha_3(x_4 - x_4^*)(x_3 - x_3^*). \quad (15)$$

Further simplification leads to the following form.

$$\frac{dV}{dt} = (k_2 c_1\alpha_1 - k_1\alpha_1)(x_1 - x_1^*)(x_2 - x_2^*) + (k_3 c_2\alpha_2 - k_2\alpha_2)(x_2 - x_2^*)(x_3 - x_3^*) + (k_4 c_3\alpha_3 - k_3\alpha_3)(x_4 - x_4^*)(x_3 - x_3^*). \quad (16)$$

Setting $k_1 = k_2 = k_3 = k_4 = 1$, we get the following form.

$$\frac{dV}{dt} = (c_1\alpha_1 - \alpha_1)(x_1 - x_1^*)(x_2 - x_2^*) + (c_2\alpha_2 - \alpha_2)(x_2 - x_2^*)(x_3 - x_3^*) + (c_3\alpha_3 - \alpha_3)(x_4 - x_4^*)(x_3 - x_3^*). \quad (17)$$

It is easy to observe that $\frac{dV}{dt} \leq 0$,

if $c_1 < 1$, $c_2 < 1$ and $c_3 < 1$. (18)

Also $\frac{dV}{dt} = 0$,

if $x_1 = x_1^*$, $x_2 = x_2^*$, $x_3 = x_3^*$ and $x_4 = x_4^*$. (19)

Hence, according to LaSalle's invariance principle, the coexistence equilibrium point is globally asymptotically stable under the mentioned condition.

Figure 3 shows the solution of systems (1-4). The values of all the parameters involved in these plots are expressed ahead $a = 0.393, c_1 = 0.51, c_2 = 0.51\ c_3 = 0.4, \alpha_1 = 0.414, \alpha_2 = 0.894, \alpha_3 = 0.9035, \alpha_4 = 0.035, \mu_1 = 0.06, \mu_2 = 0.04, \mu_3 = 0.28, \mu_4 = 0.6$the initial conditions are$(0.8, 0.2, 0.8, 0.2)$. It is obvious from the plots that the system is showing a damped oscillatory solution. The second plot shows the impact of the parameter α_2 on the population dynamics. The respective values of the parameter are 0.5943, 0.6943, and 0.7943. All the other values are taken as in Figure 4. The plots depict that rising values of the parameter raise the amplitude. Figure 5 displays the impact of the parameter c_3 on the population density. The values are taken as 0.047, 0.057, and 0.067.

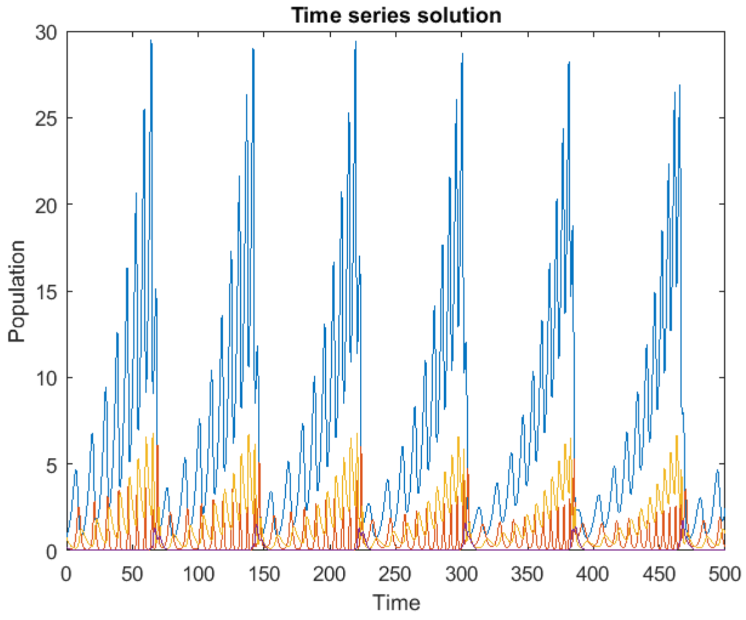
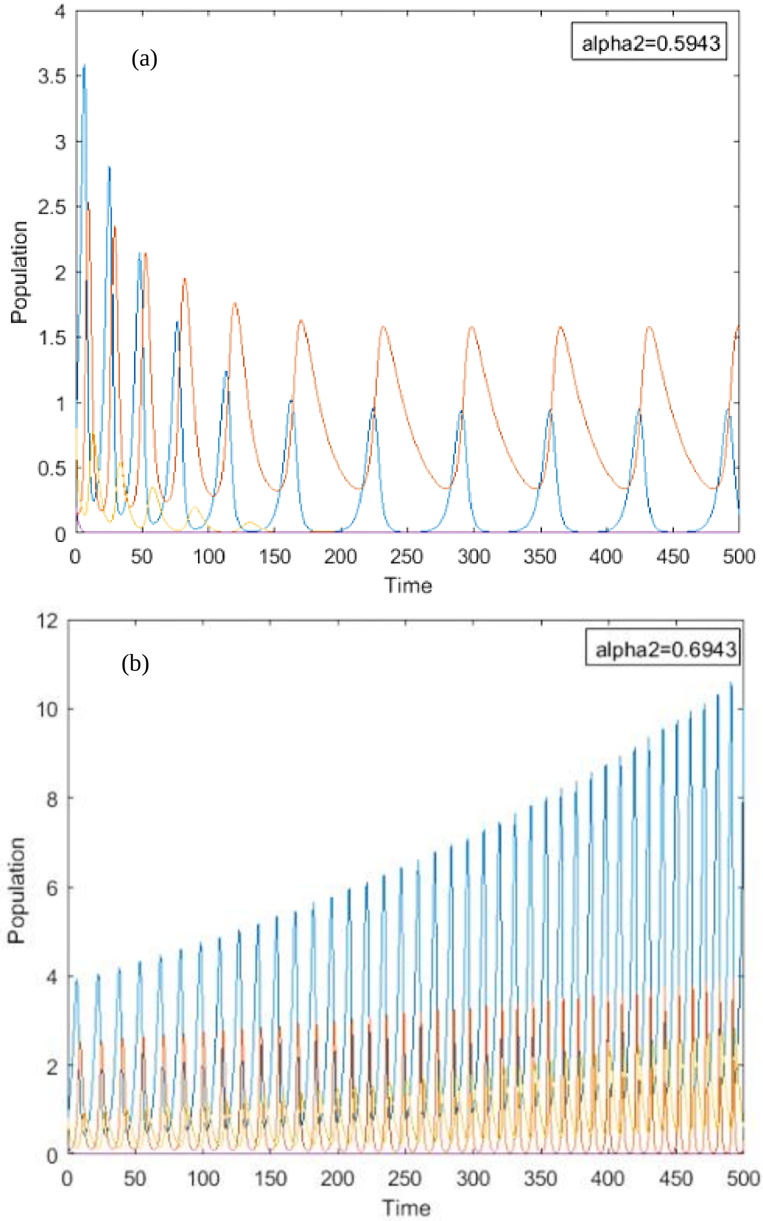


Figure 3. Solutions of the system (1-4) with initial conditions (0.8, 0.2, 0.8, 0.2)



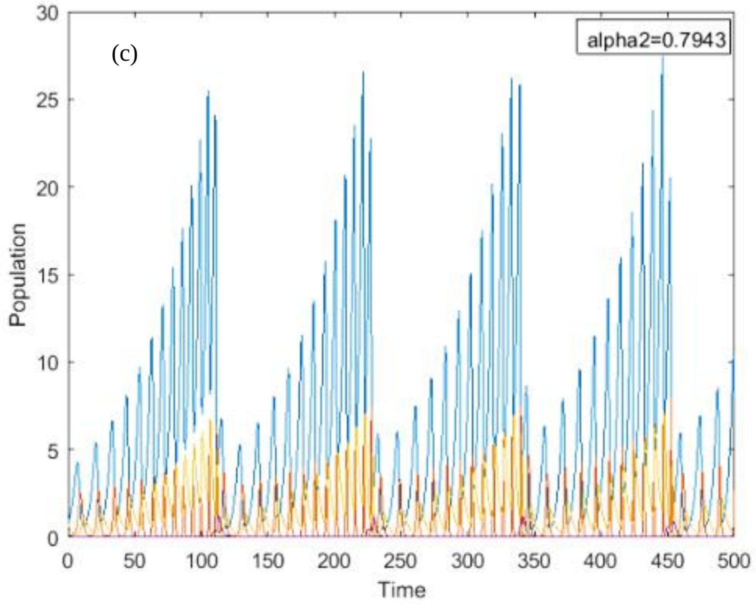
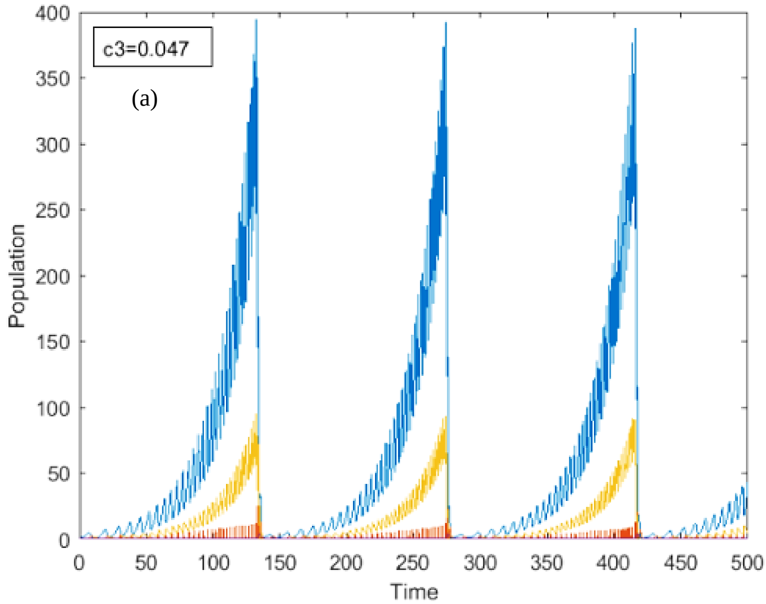
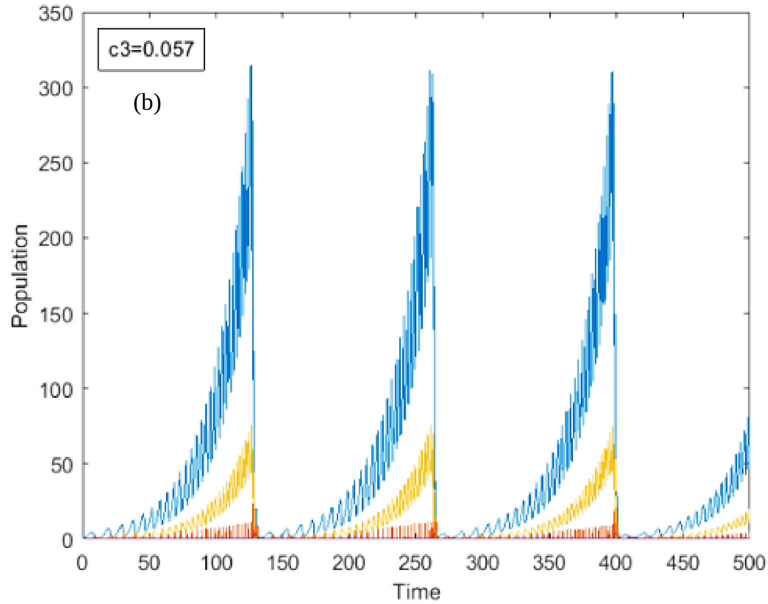


Figure 4. Impact of α_2 on population density





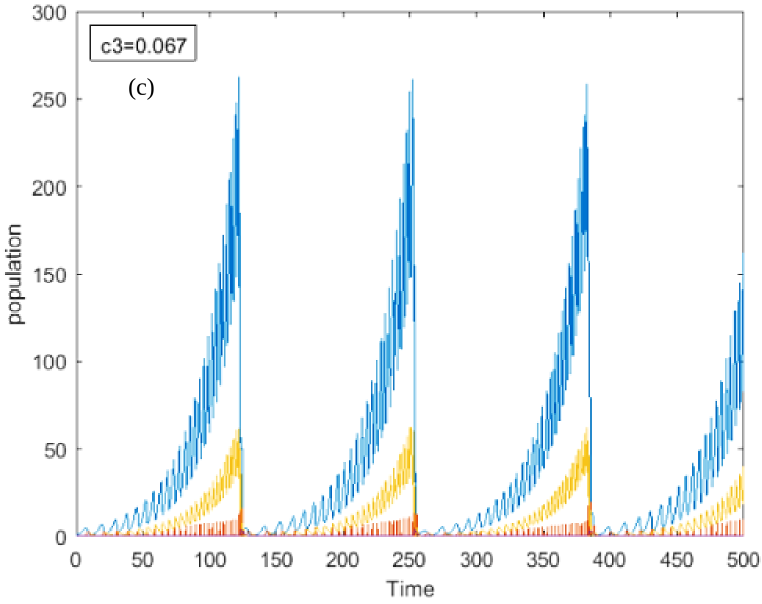
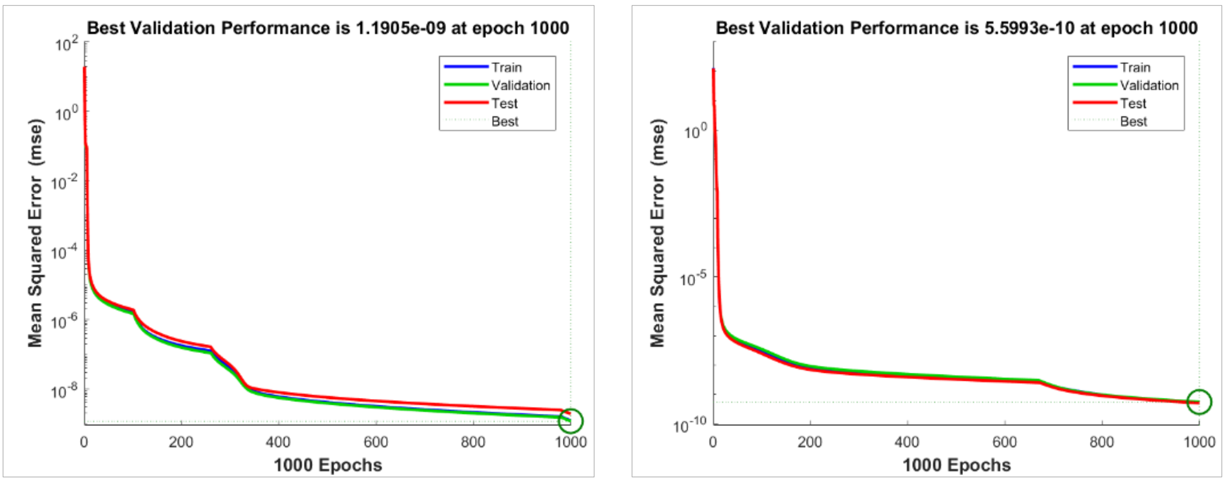


Figure 5. Impact of c_3 on population density

4- Artificial Neural Networks

Artificial Neural Networks (ANNs) are increasingly employed to model complex systems, including ecological dynamics like predator-prey interactions. ANNs simulate human brain neural networks, with artificial neurons as nodes that process and send signals across layers. Trained in historical data, ANNs can analyze population factors such as environmental conditions, species density, and interactions. This enables ANNs to predict population distributions, examine the effects of various factors, and optimize resource conservation or use. By uncovering hidden patterns in data, ANNs provide enhanced predictive accuracy and offer deeper insights into the dynamics of species interactions, particularly in systems influenced by environmental variability. In conclusion, ANNs are an effective tool for investigating predator-prey systems, improving our understanding of species dynamics and ecological stability.

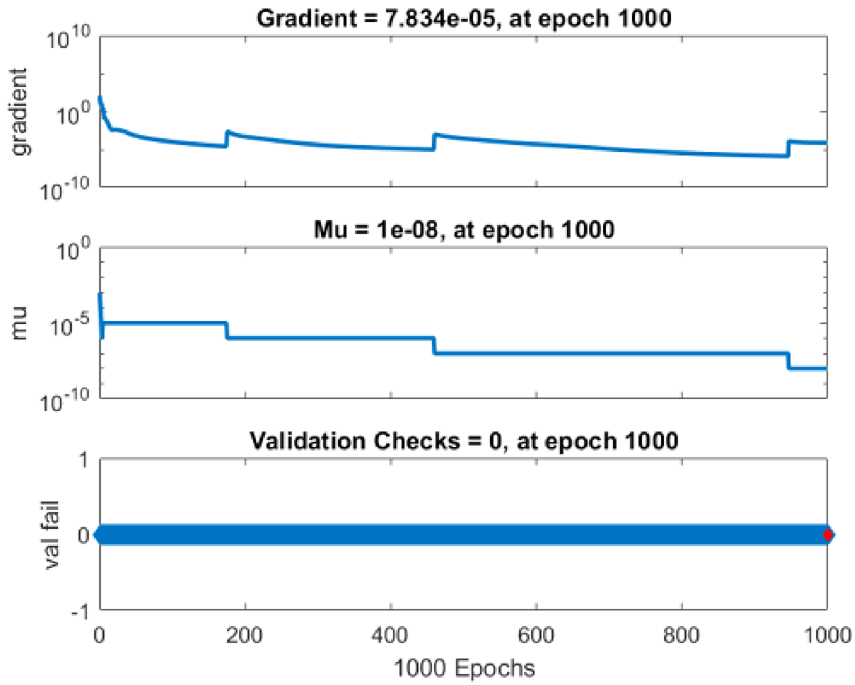
Figure 6-a plots the mean squared error of a four-species food chain model against the α_1 basal prey death rate. The artificial neural network (ANN) model epochs, or the number of times the training data is run through it, are plotted on the x-axis of the Figure. The model's fit to the training set of data is indicated by the mean squared error, which is plotted on the y-axis. The Figure shows that 1.1905×10^{-9}, which was attained at epoch 1000, had the best validation performance. This indicates that after 1000 training epochs, the model on the validation data set reached its lowest mean squared error. The plot indicates that, in general, the mean squared error decreases with increasing epochs. The mean squared error of the model is plotted against the medium predator-1 (α_2) death rate in Figure 6-b. The number of epochs, or the number of times the training data is run through the artificial neural network (ANN) model, is represented on the x-axis. The mean squared error, a gauge of how well the model matches the training set of data, is displayed on the y-axis. The best validation performance, 5.5993×10^{-10}, was attained at epoch 1000. This indicates that after 1000 training epochs, the model reached its lowest mean squared error on the validation data set.



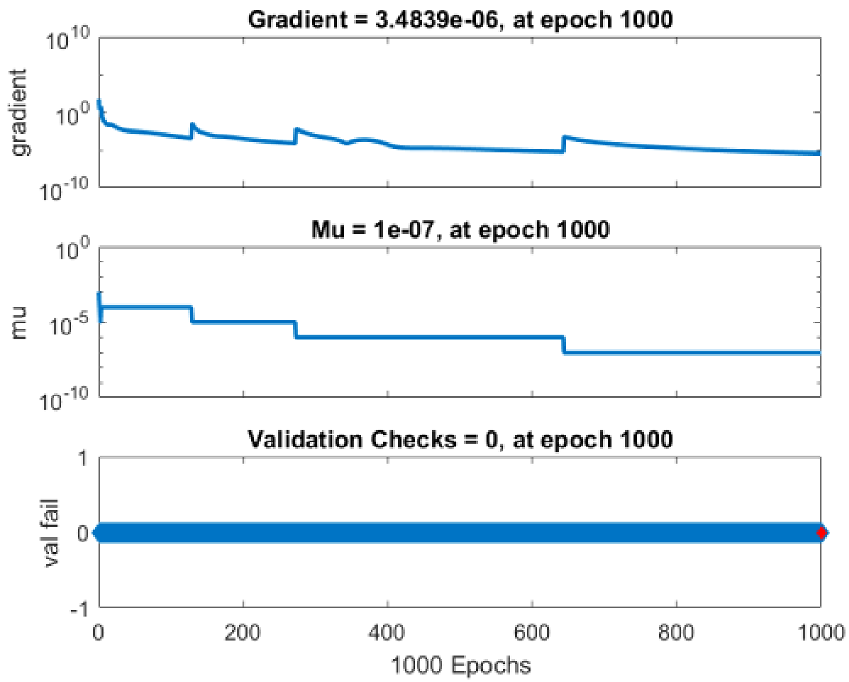
(a) (b)

Figure 6. Training process of artificial neural network (ANN) for the impact of (a) α1 (b) α2

Figure 7-a shows the results of the transition state for α_1, focusing on gradient and validation checks. The number of epochs, or the number of times the training data is run through the artificial neural network (ANN) model, is represented on the x-axis. The graphic shows that during the 1000 epochs, there were no validation checks. Two values for the gradient are mentioned in the text annotations: Mu 1×10^{-8} at epoch 1000 and 7.834×10^{-5} at epoch 1000. The gradient indicates how much the weights in the model should be changed to enhance the fit between the predicted and observed data. On the whole, the plot suggests that the model achieved a validation check value of zero. In Figure 7-b, plotted for α_2, two values for the gradient are mentioned in the text annotations: Mu 1×10^{-7} at epoch 1000 and 3.4839× 10^{-6} at epoch 1000. The gradient indicates how much the weights in the model should be changed to enhance the fit between the predicted and observed data. To put it another way, it shows which way the error function's steepest descent is. The model may be approaching a minimum error state if the gradient has a smaller value. In this case, the gradient exhibits a positive sign at epoch 1000 and looks to be quite modest.



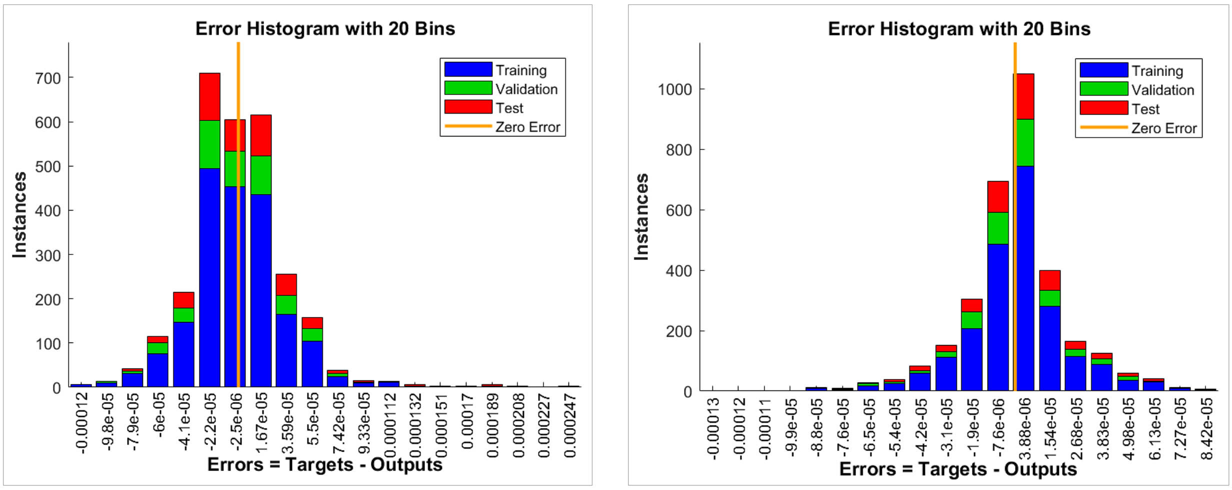
(a)



(b)

Figure 7. Results of the transition state on the impact of (a) α_1 (b) α_2

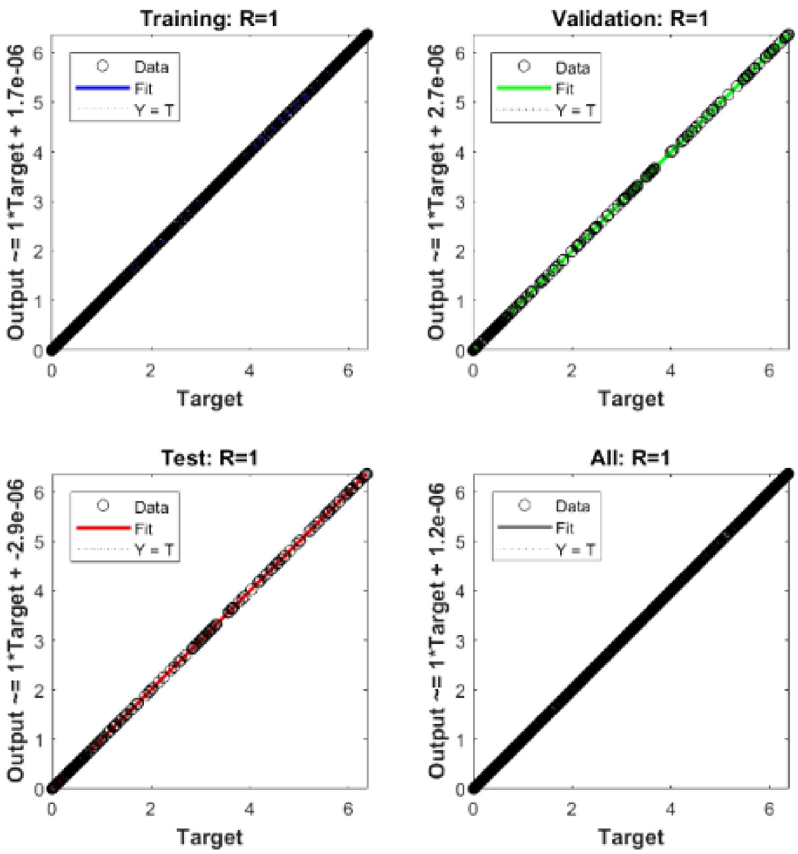
The error distribution between the goal values and the output values of an ANN model for estimating the cannibalism rate of basal prey is represented visually in error histograms Figure 8(a) and (b). The histogram is divided into 20 bins, where the y-axis displays the number of occurrences inside each bin, and the x-axis represents the range of errors. The majority of the errors are centered on zero, demonstrating the effectiveness of the ANN model. On the left and right, there is a tiny tail of errors, though, which suggests that the actual values deviate significantly from the goal values.



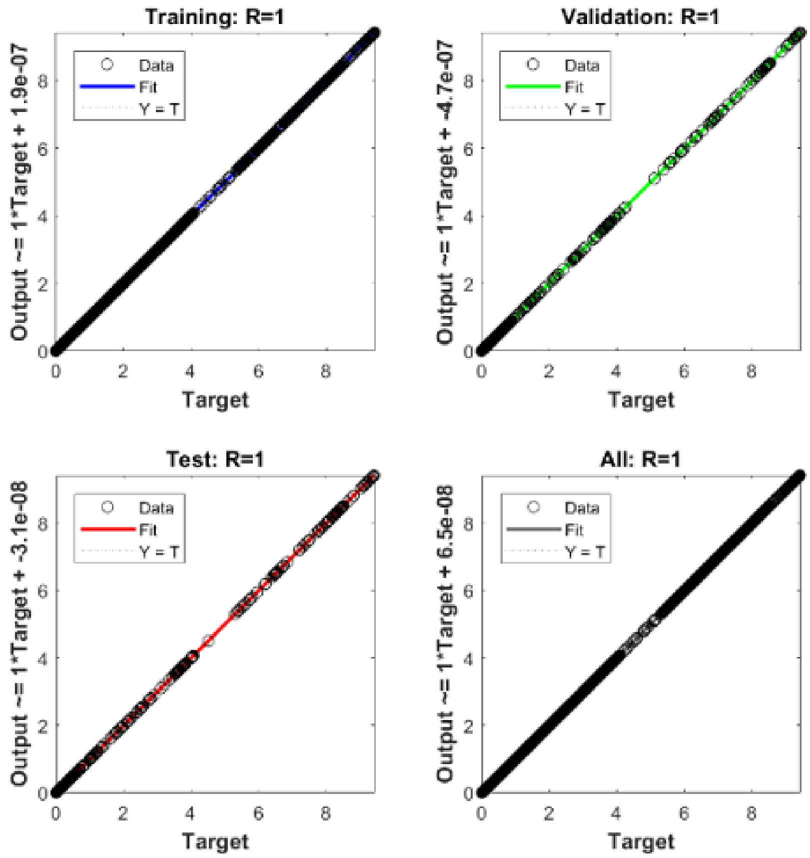
(a) (b)

Figure 8. Histogram of error analysis for the impact of (a) α_1 (b) α_2

Four graphs pertaining to the training and validation of a neural network for regression analysis are displayed in Figures 9-a and 9-b. All of the data points in the top left graph, which depicts the connection between the actual and anticipated goal values during the training phase, lie on the diagonal line. During the validation phase, the connection between the predicted and actual target values is depicted in the top right graph, where all data points are in close proximity to the diagonal line. On a different test dataset, the link between projected and actual target values is displayed in the bottom left graph, where all data points are near the diagonal line. The bottom right graph shows a strong overall fit of the neural network by combining data from all three phases into a single figure.



(a)



(b)

Figure 9. Regression analysis from using the target of the impact of (a) α_1 (b) α_2

5- Diffusion System

Diffusion is justified in predator-prey models because predators and prey are not equally distributed. Instead, they may cluster or disperse, causing regional variation in the two species population dynamics. Diffusion mathematically models spatial heterogeneity by including random population movements. Diffusion can reflect predator-prey mobility through their shared environment and predator-prey interactions in predator-prey models. Diffusion in predator-prey models can also explain how predator and prey populations can become patchily dispersed over time, with high and low densities. This may affect the two species' survival and environmental interactions. Diffusion in predator-prey models can assist explains the intricate dynamics of these ecological relationships and reveal the principles that sustain ecosystems.

In ecological models, diffusion deals with the spatial spread of populations, where species disperse and interact based on their locations. Generally, in a food chain model, diffusion describes how prey species may spread across different regions, influencing their interaction with other species in the system. In contrast, non-diffusion factors involve internal processes like predation, birth, or death rates, which do not depend on spatial distribution. These non-diffusion factors capture interactions such as predation dynamics or competition within a population. Both diffusion and non-diffusion factors are crucial for modeling species interactions in food chains because they affect population dynamics through different mechanisms.

One common mathematical representation of diffusion in predator-prey models is the use of partial differential equations to depict the temporal and spatial variation in predator and prey density. These equations account for the velocities of people in the environment and their interactions with one another.

We examine the impact of diffusion on the density of interacting species' populations using the self-diffusive predator-prey model. We take d_i, $i = 1,2,3,4$ as diffusion coefficient.

The system defined in (1-4) takes the following form.

$$\frac{dx_1}{dt} = d_1\frac{\partial^2 x_1}{\partial x^2} + x_1(a - x_2\alpha_1 - \mu_1), \qquad t > 0, x > 0 \tag{20}$$

$$\frac{dx_2}{dt} = d_2\frac{\partial^2 x_2}{\partial x^2} + x_2(c_1x_1\alpha_1 - x_3\alpha_2 - \mu_2), \qquad t > 0, x > 0 \tag{21}$$

$$\frac{dx_3}{dt} = d_3\frac{\partial^2 x_3}{\partial x^2} + x_3(c_2x_2\alpha_2 - x_4\alpha_3 - \mu_3)\,, \qquad t > 0, x > 0 \tag{22}$$

$$\frac{dx_4}{dt} = d_4 \frac{\partial^2 x_4}{\partial x^2} + x_4(c_3 x_3 \alpha_3 - \mu_4), t > 0, \qquad x > 0 \tag{23}$$

where;

$x_1(0,0) \geq 0, x_2(0,0) \geq 0, x_3(0,0) \geq 0, x_4(0,0) \geq 0.$

and Neumann boundary conditions

$$\frac{\partial x_1}{\partial \nu} = \frac{\partial x_2}{\partial \nu} = \frac{\partial x_3}{\partial \nu} = \frac{\partial x_4}{\partial \nu} = 0, \qquad x \in \partial\Omega.$$

Figure 10 shows contour plots for the diffusion model. The parameters in these plots are taken as $a = 0.933, c_1 = 0.848$, $c_2 = 0.008$, $c_3 = 0.0014$, $\alpha_1 = 0.165$, $\alpha_2 = 0.143$, $\alpha_3 = 0.199$, $\alpha_4 = 0.108$, $\mu_1 = 0.06$, $\mu_2 = 0.604$, $\mu_3 = 0.028$, $\mu_4 = 0.6, d_1 = 0.804, d_2 = 0.93, d_3 = 0.007, d_4 = 0$. Typically, contour maps are used to visualize the distribution of variables under study across a two-dimensional space. In the present case, the variable being mapped is the concentration or density of each species in the food chain diffusion model. The red strips likely represent areas of high concentration or density for the species in the food chain. The narrower width of the strips could indicate that the high-concentration areas are more localized or concentrated in specific areas. The green strips that are wider than the red ones indicate areas where the respective species in the food chain is more concentrated or abundant. Overall, the different colored strips on the contour map provide a visual representation of the distribution of species in the food chain diffusion model and can help to identify areas of high and low concentration or density for each species.

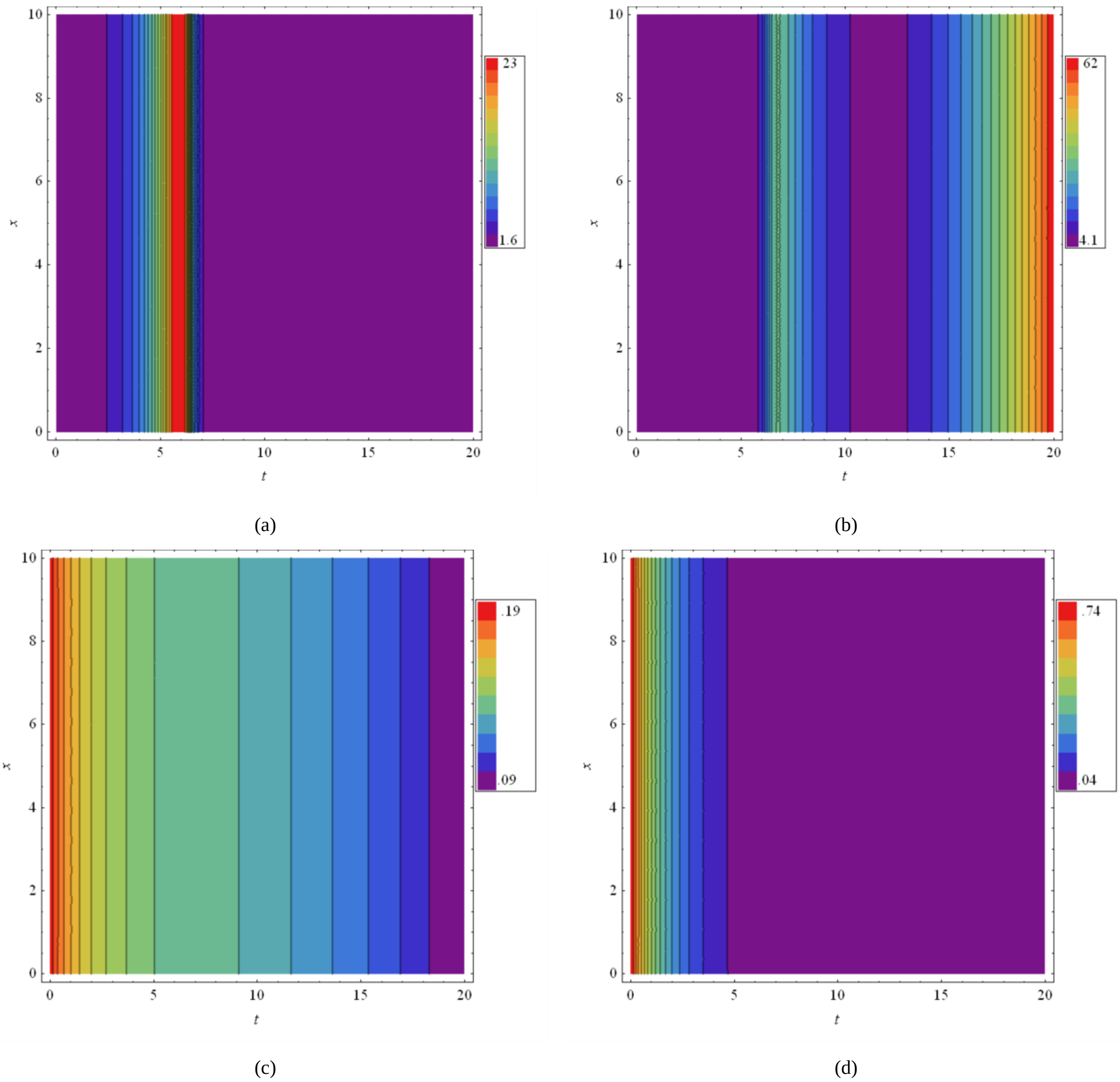


Figure 10. Contour plots for the diffusion system with ICs (0.2, 0.8, 0.2, 0.8)

6- Numerical Scheme

In this section, we construct a numerical scheme using three-time levels.

6-1- Construction of Numerical Scheme

To construct the scheme, suppose we have the following diffusion equation.

$$\frac{\partial w}{\partial t} = d_1 \frac{\partial^2 w}{\partial^2 x} + \bar{\alpha} w$$

We construct an implicit scheme by considering the following difference equation. In this construction, three-time levels are taken, i.e., $n,\ n+1\ and\ n-1$.

$$w_i^{n+1} = a w_i^n + 3\frac{v_i^{n-1}}{4} + \Delta t \left\{ b\left(\frac{\partial w}{\partial t}\right)_i^{n+1} + c\left(\frac{\partial w}{\partial t}\right)_i^{n} + e\left(\frac{\partial w}{\partial t}\right)_i^{n-1} \right\} \tag{24}$$

The Taylor series expansions for $w_i^{n+1}, w_i^{n-1}, \left(\frac{\partial w}{\partial t}\right)_i^{n+1}$ and $\left(\frac{\partial w}{\partial t}\right)_i^{n-1}$ are given as,

$$w_i^{n+1} = w_i^n + \Delta t\left(\frac{\partial w}{\partial t}\right)_i^n + \frac{(\Delta t)^2}{2}\left(\frac{\partial^2 w}{\partial t^2}\right)_i^n + \frac{(\Delta t)^3}{6}\left(\frac{\partial^3 w}{\partial t^3}\right)_i^n + O((\Delta t)^4), \tag{25}$$

$$w_i^{n-1} = w_i^n - \Delta t\left(\frac{\partial w}{\partial t}\right)_i^n + \frac{(\Delta t)^2}{2}\left(\frac{\partial^2 w}{\partial t^2}\right)_i^n - \frac{(\Delta t)^3}{6}\left(\frac{\partial^3 w}{\partial t^3}\right)_i^n + O((\Delta t)^4), \tag{26}$$

$$\left(\frac{\partial w}{\partial t}\right)_i^{n+1} = \left(\frac{\partial w}{\partial t}\right)_i^n + \Delta t\left(\frac{\partial^2 w}{\partial t^2}\right)_i^n + \frac{(\Delta t)^2}{2}\left(\frac{\partial^3 w}{\partial t^3}\right)_i^n + O((\Delta t)^3), \tag{27}$$

$$\left(\frac{\partial w}{\partial t}\right)_i^{n-1} = \left(\frac{\partial w}{\partial t}\right)_i^n - \Delta t\left(\frac{\partial^2 w}{\partial t^2}\right)_i^n + \frac{(\Delta t)^2}{2}\left(\frac{\partial^3 w}{\partial t^3}\right)_i^n - O((\Delta t)^3). \tag{28}$$

Putting the above values in the above equation and further simplification leads to the following equations.

$$1 = a + \frac{3}{4}, \tag{29}$$

$$1 = -\frac{3}{4} + b + c + e, \tag{30}$$

$$\frac{1}{2} = \frac{3}{8} + b - e, \tag{31}$$

$$\frac{1}{6} = -\frac{1}{8} + \frac{b}{2} + \frac{e}{2}. \tag{32}$$

Solving Equations 29 to 32 gives the values of unknown parameters $a, b, c,$ and e as given below.

$$a = \frac{1}{4}, b = \frac{17}{48}, c = \frac{7}{6}, e = \frac{11}{48}. \tag{33}$$

So, we have the following form.

$$w_i^{n+1} = \frac{1}{4} w_i^n + 3\frac{w_i^{n-1}}{4} + \Delta t\left\{\frac{17}{48}\left(\frac{\partial w}{\partial t}\right)_i^{n+1} + \frac{7}{6}\left(\frac{\partial w}{\partial t}\right)_i^{n} + \frac{11}{48}\left(\frac{\partial w}{\partial t}\right)_i^{n-1}\right\}$$

Further simplification leads to the following result.

$$w_i^{n+1} = \frac{1}{4}(w_i^n + 3w_i^{n-1}) + \frac{\Delta t}{48}\left\{17\left(\frac{\partial w}{\partial t}\right)_i^{n+1} + 56\left(\frac{\partial w}{\partial t}\right)_i^{n} + 11\left(\frac{\partial w}{\partial t}\right)_i^{n-1}\right\} \tag{34}$$

6-2- Stability Analysis

We use the Von Neumann stability technique to investigate the stability of the constructed numerical scheme. For applying this technique, we consider the following diffusive linear predator-prey equation:

$$\frac{\partial w}{\partial t} = d_1 \frac{\partial^2 w}{\partial^2 x} + \bar{\alpha} w. \tag{35}$$

The discretization of Equation 35 using the presented scheme is given as under.

$$w_i^{n+1} = a w_i^n + \frac{3}{4} w_i^{n-1} + \Delta t\left\{ b d_1\left(\frac{w_{i+1}^{n+1} - 2w_i^{n+1} + w_{i-1}^{n+1}}{(\Delta x)^2}\right) + b\bar{\alpha} w_i^{n+1} + c d_1\left(\frac{w_{i+1}^{n} - 2w_i^{n} + w_{i-1}^{n}}{(\Delta x)^2} + \bar{\alpha} w_i^n\right) + c\bar{\alpha} w_i^n + e d_1\left(\frac{w_{i+1}^{n-1} - 2w_i^{n-1} + w_{i-1}^{n-1}}{(\Delta x)^2} + e\bar{\alpha} w_i^{n-1}\right)\right\}. \tag{36}$$

We consider the following transformations for analysis.

$$w_i^n = A^n e^{iI\phi}, w_i^{n+1} = A^{n+1} e^{iI\phi}, w_{i\pm1}^n = A^n e^{(i\pm1)I\phi}, w_{i\pm1}^{n-1} = A^{n+1} e^{(i\pm1)I\phi} \tag{37}$$

Here, $I = \sqrt{-1}$.

Substituting the above values in Equation 36 and multiplying both sides by $e^{-iI\phi}$ yields the following form.

$$A^{n+1} = aA^n + \frac{3}{4}A^{n-1} + \Delta t \begin{Bmatrix} bd_1\left(\frac{2cos\phi-2}{(\Delta x)^2}\right)A^{n+1} + b\bar{\alpha}A^{n+1} + \\ cd_1\left(\frac{2cos\phi-2}{(\Delta x)^2}\right)A^n + c\bar{\alpha}A^n + \\ ed_1\left(\frac{2cos\phi-2}{(\Delta x)^2}\right)A^{n-1} + e\bar{\alpha}A^{n-1} \end{Bmatrix} \tag{38}$$

Taking $d = \frac{\Delta t}{(\Delta x)^2}$ and collecting coefficients of A^{n+1} on the left-hand side provides the form given as under.

$$(1 - 2bdd_1(cos\phi - 1) - b\Delta t\bar{\alpha})A^{n+1} = aA^n + \frac{3}{4}A^{n-1} + 2cdd_1(cos\phi - 1)A^n + \Delta t\bar{\alpha}A^{n+1} + 2edd_1(cos\phi - 1)A^{n-1} + e\Delta t\bar{\alpha}A^{n-1}. \tag{39}$$

Equation 38 can be written as below:

$$A^{n+1} = PA^n + QA^{n-1} \tag{40}$$

where,

$$P = \frac{a+2cdd_1(cos\phi-1)+c\Delta t\bar{\alpha}}{1-2bdd_1(cos\phi-1)-b\Delta t\bar{\alpha}} \text{ and } Q = \frac{3+8edd_1(cos\phi-1)+4e\Delta t\bar{\alpha}}{4(1-2bdd_1(cos\phi-1)-b\Delta t\bar{\alpha})}. \tag{41}$$

To write the amplification matrix, we need another equation. The additional equation is expressed below.

$$A^n = 1.A^n + 0.A^{n-1} \tag{42}$$

The matrix-vector equation can be expressed as below:

$$\begin{bmatrix} A^{n+1} \\ A^n \end{bmatrix} = \begin{bmatrix} P & Q \\ 1 & 0 \end{bmatrix} \begin{bmatrix} A^n \\ A^{n-1} \end{bmatrix}. \tag{43}$$

The stability criteria for the amplification matrix's eigenvalues are as follows:

$$\left|\frac{P+\sqrt{P^2+4Q}}{2}\right| \le 1 \text{ and } \left|\frac{P-\sqrt{P^2+4Q}}{2}\right| \le 1\ . \tag{44}$$

If the eigenvalues of the amplification matrix are positive, then the scheme is stable if the inequalities (44) are satisfied, and if the eigenvalues are negative, then the stability requirement is expressed as:

$$|Q|^2 \le 1. \tag{45}$$

where $P\ and\ Q$ are given in Equation 41.

6-3- Numerical Simulations

We apply the constructed third-order multistep finite difference scheme to the model presented in Equations 20-23.

$$v_i^{n+1} = \frac{1}{4}(v_i^n + 3v_i^{n-1}) + \frac{\Delta t}{48}\left\{17\left(\frac{\partial v}{\partial t}\right)_i^{n+1} + 56\left(\frac{\partial v}{\partial t}\right)_i^n + 11\left(\frac{\partial v}{\partial t}\right)_i^{n-1}\right\}$$
$$x_{1,i}^{n+1} = \frac{1}{4}(x_{1,i}^n + 3x_{1,i}^{n-1}) + \frac{\Delta t}{48}\left\{17\left(\frac{\partial x_1}{\partial t}\right)_i^{n+1} + 56\left(\frac{\partial x_1}{\partial t}\right)_i^n + 11\left(\frac{\partial x_1}{\partial t}\right)_i^{n-1}\right\}.$$

The above equation can be written as follows:

$$x_{1,i}^{n+1} = \frac{1}{4}(x_{1,i}^n + 3x_{1,i}^{n-1}) + \frac{\Delta t}{48}\begin{bmatrix} 17\begin{Bmatrix} d_1\left(x_{1,i+1}^{n+1} - 2x_{1,i}^{n+1} + x_{1,i-1}^{n+1}\right)/(\Delta x)^2 + \\ x_{1,i}^{n+1}(a - x_{2,i}^{n+1}\alpha_1 - \mu_1) \end{Bmatrix} + \\ 56\begin{Bmatrix} d_1\left(x_{1,i+1}^{n} - 2x_{1,i}^{n} + x_{1,i-1}^{n}\right)/(\Delta x)^2 + \\ x_{1,i}^{n}(a - x_{2,i}^{n}\alpha_1 - \mu_1) \end{Bmatrix} + 11 \\ \begin{Bmatrix} d_1\left(x_{1,i+1}^{n-1} - 2x_{1,i}^{n-1} + x_{1,i-1}^{n-1}\right)/(\Delta x)^2 + \\ x_{1,i}^{n-1}(a - x_{2,i}^{n-1}\alpha_1 - \mu_1) \end{Bmatrix} \end{bmatrix} \tag{46}$$

Similarly, the other equations of the system described above are as follows:

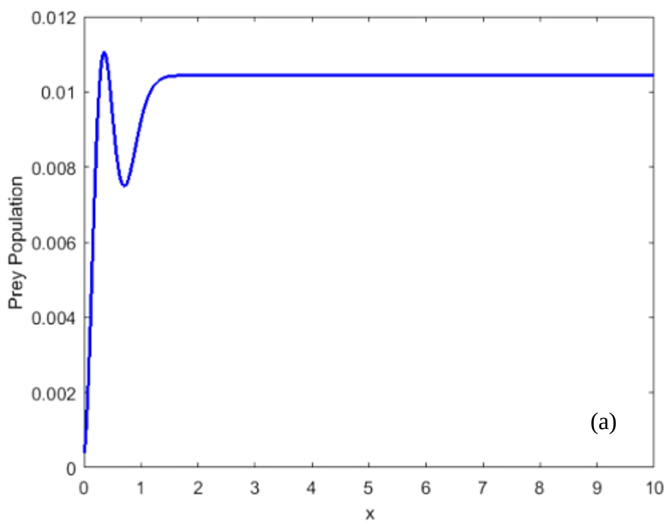
$$x_{2,i}^{n+1}=\frac{1}{4}\left(x_{2,i}^{n}+3x_{2,i}^{n-1}\right)+\frac{\Delta t}{48}\left[\begin{array}{c}17\left\{\begin{array}{c}d_2\left(x_{2,i+1}^{n+1}-2x_{2,i}^{n+1}+x_{2,i-1}^{n+1}\right)/(\Delta x)^2+\\ x_{2,i}^{n+1}\left(c_1x_{1,i}^{n+1}\alpha_1-x_{3,i}^{n+1}\alpha_2-\mu_2\right)\end{array}\right\}+\\ 56\left\{\begin{array}{c}d_2\left(x_{2,i+1}^{n}-2x_{2,i}^{n}+P_{2,i-1}^{n}\right)/(\Delta x)^2+\\ x_{2,i}^{n}\left(c_1x_{1,i}^{n}\alpha_1-x_{3,i}^{n}\alpha_2-\mu_2\right)\end{array}\right\}+11\\ \left\{\begin{array}{c}d_2\left(x_{2,i+1}^{n-1}-2x_{2,i}^{n-1}+x_{2,i-1}^{n-1}\right)/(\Delta x)^2+\\ x_{2,i}^{n-1}\left(c_1x_{1,i}^{n-1}\alpha_1-x_{3,i}^{n-1}\alpha_2-\mu_2\right)\end{array}\right\}\end{array}\right] \tag{47}$$

$$x_{3,i}^{n+1}=\frac{1}{4}\left(x_{3,i}^{n}+3x_{3,i}^{n-1}\right)+\frac{\Delta t}{48}\left[\begin{array}{c}17\left\{\begin{array}{c}d_3\left(x_{3,i+1}^{n+1}-2x_{3,i}^{n+1}+x_{3,i-1}^{n+1}\right)/(\Delta x)^2+\\ x_{3,i}^{n+1}\left(c_2x_{3,i+1}^{n+1}\alpha_2-x_{3,i+1}^{n+1}\alpha_3-\mu_3\right)\end{array}\right\}+\\ 56\left\{\begin{array}{c}d_3\left(x_{3,i+1}^{n}-2x_{3,i}^{n}+x_{3,i-1}^{n}\right)/(\Delta x)^2+\\ x_{3,i}^{n}\left(c_2x_{3,i+1}^{n}\alpha_2-x_{3,i+1}^{n}\alpha_3-\mu_3\right)\end{array}\right\}+11\\ \left\{\begin{array}{c}d_3\left(x_{3i+1}^{n-1}-2x_{3,i}^{n-1}+x_{3,i-1}^{n-1}\right)/(\Delta x)^2+\\ x_{3,i}^{n-1}\left(c_2x_{3,i+1}^{n-1}\alpha_2-x_{3,i+1}^{n-1}\alpha_3-\mu_3\right)\end{array}\right\}\end{array}\right] \tag{48}$$

$$x_{4,i}^{n+1}=\frac{1}{4}\left(x_{4,i}^{n}+3x_{4,i}^{n-1}\right)+\frac{\Delta t}{48}\left[\begin{array}{c}17\left\{\begin{array}{c}d_4\left(x_{4,i+1}^{n+1}-2x_{4,i}^{n+1}+x_{4,i-1}^{n+1}\right)/(\Delta x)^2+\\ x_{4,i}^{n+1}\left(c_3x_{3,i+1}^{n+1}\alpha_3-\mu_4\right)\end{array}\right\}+\\ 56\left\{\begin{array}{c}d_3\left(x_{4,i+1}^{n}-2x_{4,i}^{n}+x_{4,i-1}^{n}\right)/(\Delta x)^2+\\ x_{4,i}^{n}\left(c_3x_{3,i+1}^{n}\alpha_3-\mu_4\right)\end{array}\right\}+11\\ \left\{\begin{array}{c}d_3\left(x_{4,i+1}^{n-1}-2x_{4,i}^{n-1}+x_{4,i-1}^{n-1}\right)/(\Delta x)^2+\\ x_{4,i}^{n-1}\left(c_3x_{3,i+1}^{n-1}\alpha_3-\mu_4\right)\end{array}\right\}\end{array}\right] \tag{49}$$

The above Equations 46 to 49 describe the diffusive system's discretization Equations 20-23. The graphical presentation of the system (20-23) through the presented scheme is as follows.

Figure 11 presents the solution of the diffusive system for the species population. The diffusion coefficients are taken as $(0.98, 0.999, 0.007, 0.15)$. The values of other parameters are $a = 0.7, c_1 = 0.848, c_2 = 0.008, c_3 = 0.0014, \alpha_1 = 0.16, \alpha_2 = 0.843, \alpha_3 = 0.19, \mu_1 = 0.006, \mu_2 = 0.604, \mu_3 = 0.028, \mu_4 = 0.909.$ Figure 12 shows the solution of the diffusive system for the parameter values as in Figure 3. Figure 13 shows the bifurcation plots for all the populations by taking $'a'$ as the bifurcation parameter. The values of $'a'$ are taken up to 0.4. The values of parameters in these plots are $a = 0.933, c_1 = 0.88, c_2 = 0.999\ c_3 = 0.14, \alpha_1 = 0.9916, \alpha_2 = 0.4814, \alpha_3 = 0.4519, \mu_1 = 0.006, \mu_2 = 0.604, \mu_3 = 0.0028, \mu_4 = 0.03239$, with initial condition$(0.9, 0.19, 0.7, 0.8)$. Figure 14 depicts the comparison of solutions for all the interactive species for two distinct values of respective diffusion coefficients. Here parameters are $d_1 = (0.68, 0.78), d_2 = (0.899, 0.999), d_3 = (0.005, 0.007), d_4 = (0.19, 0.29)$. The rest of the parameters are the same as those in Figure 11. The analysis underscores the critical influence of diffusion parameters on species population dynamics within the system. Increasing the diffusion values generally exerts inverse effects on the population densities of all species, except for medium predator-2, which exhibits a direct correlation with diffusion rates. This distinction highlights the unique ecological interactions and adaptive strategies of medium predator-2, potentially reflecting its pivotal role in the trophic hierarchy. Figure 14 illustrates the impact of migration parameters on the prey population, showing an inverse relationship. As migration rates enhance, prey populations decrease, suggesting that elevated migration moves to a more uniform spatial distribution and decreases localized population densities. This result highlights the role of spatial heterogeneity and migration in regulating prey dynamics. Overall, these results emphasize the complex relationship between diffusion and migration in shaping ecological systems.



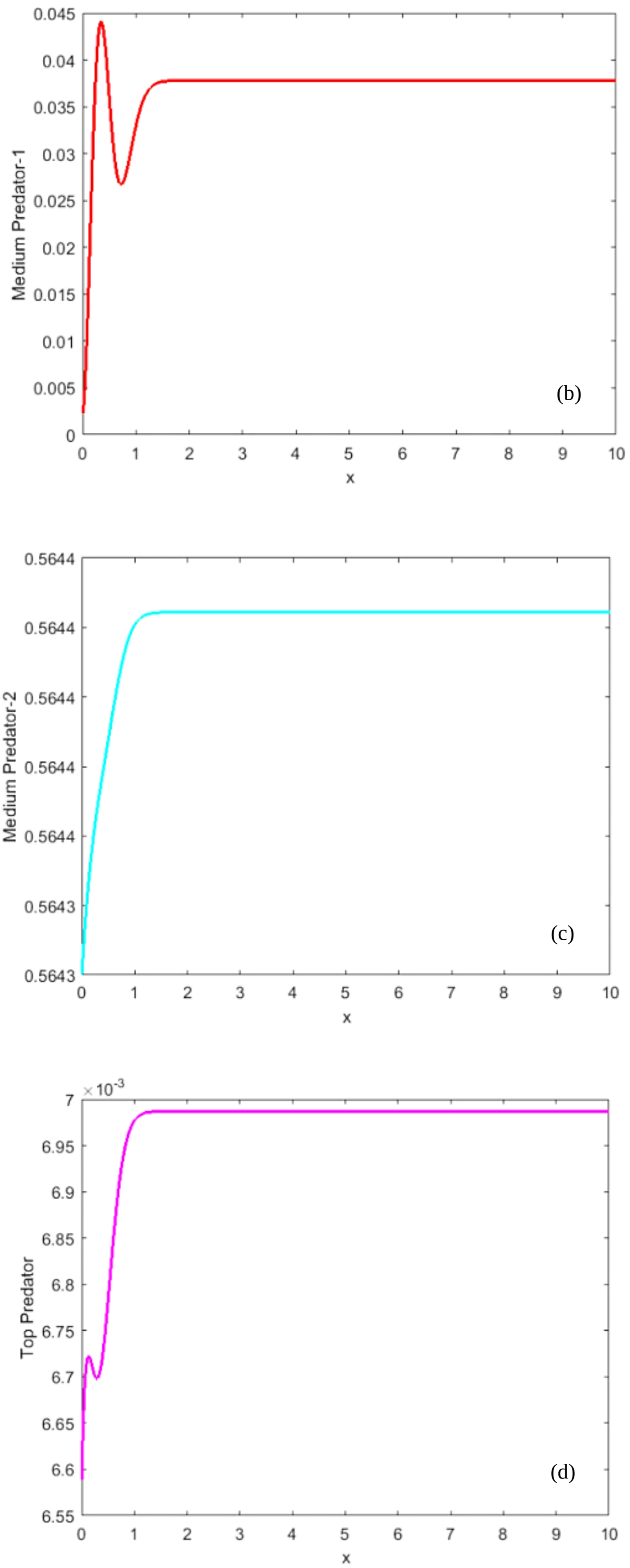
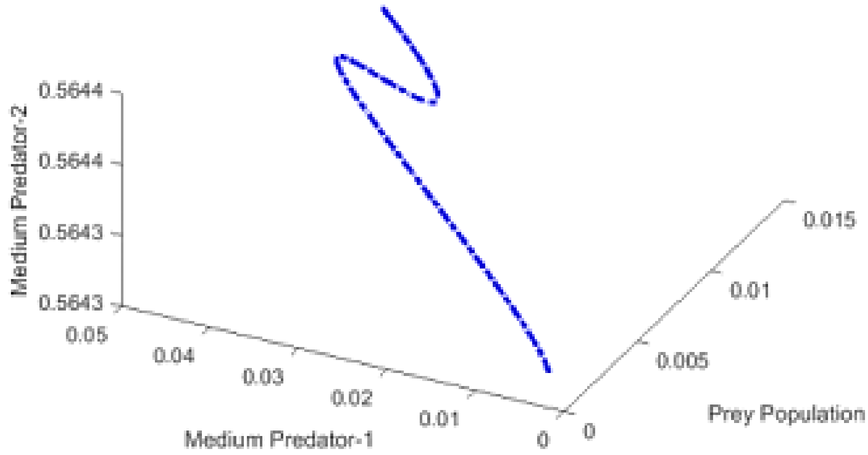
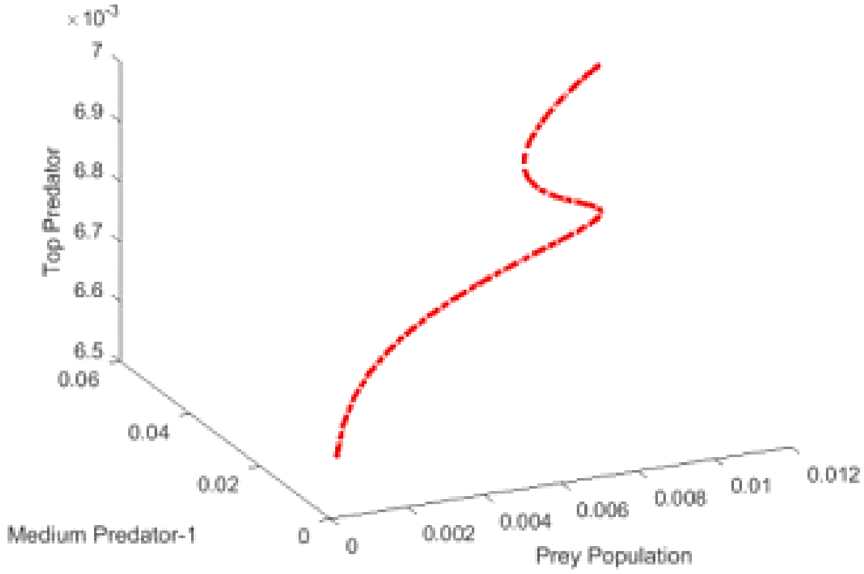


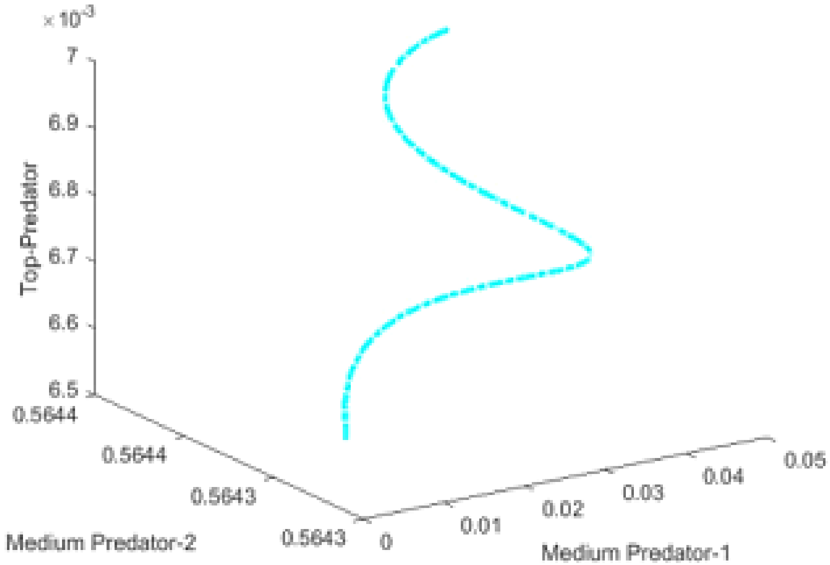
Figure 11. **Solution for the diffusive system with initial conditions (0.17, 0.92. 0.58, 0.02)**



(a)

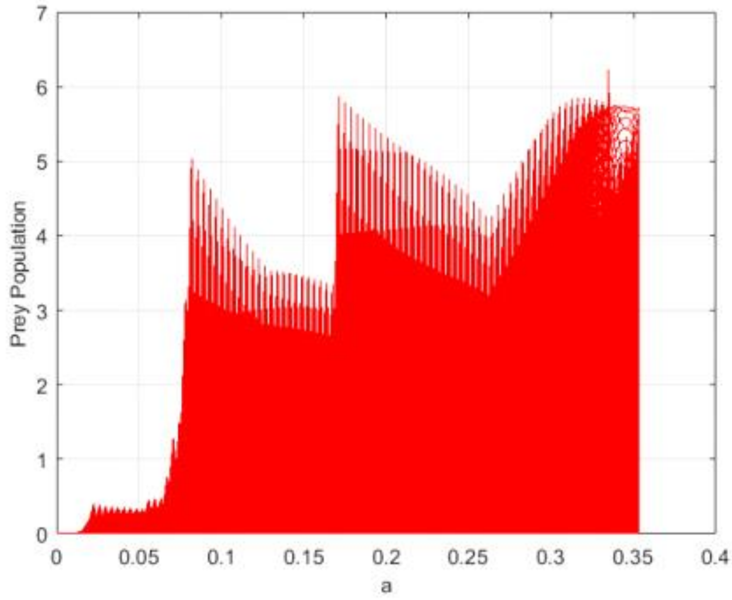


(b)

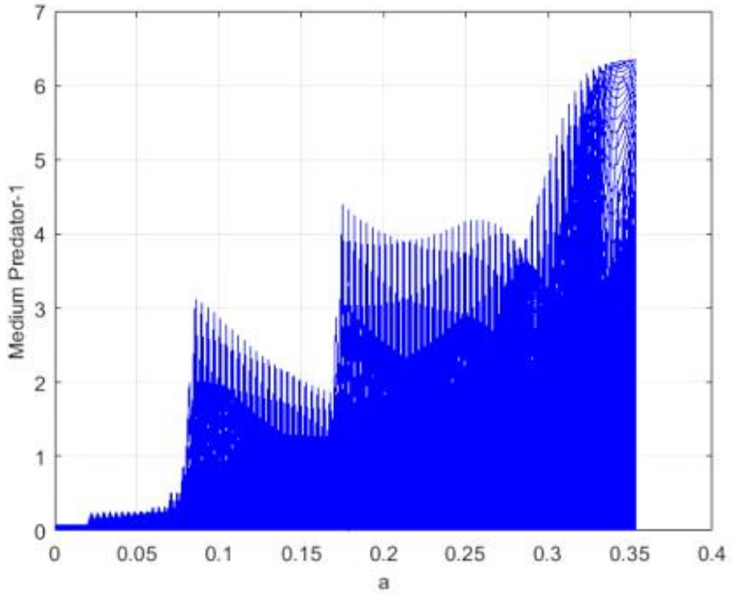


(c)

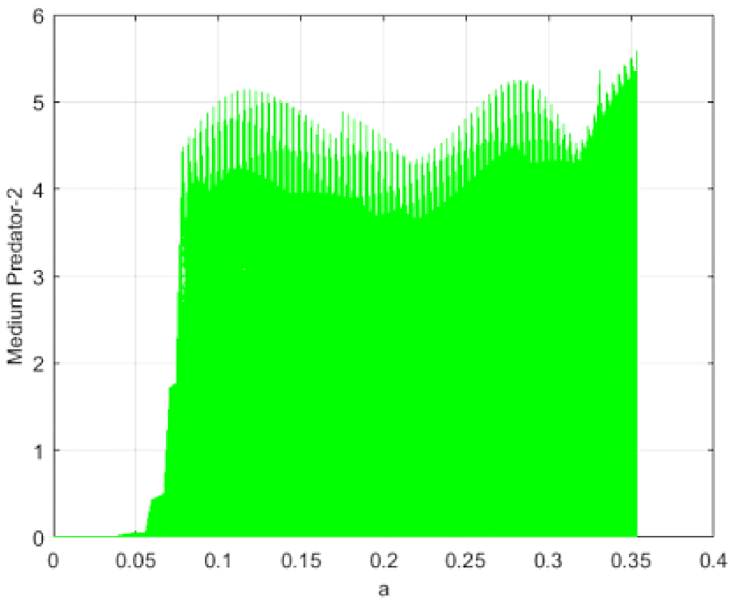
Figure 12. Numerical solution for the diffusive system


Prey Population
a
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0
1
2
3
4
5
6
7

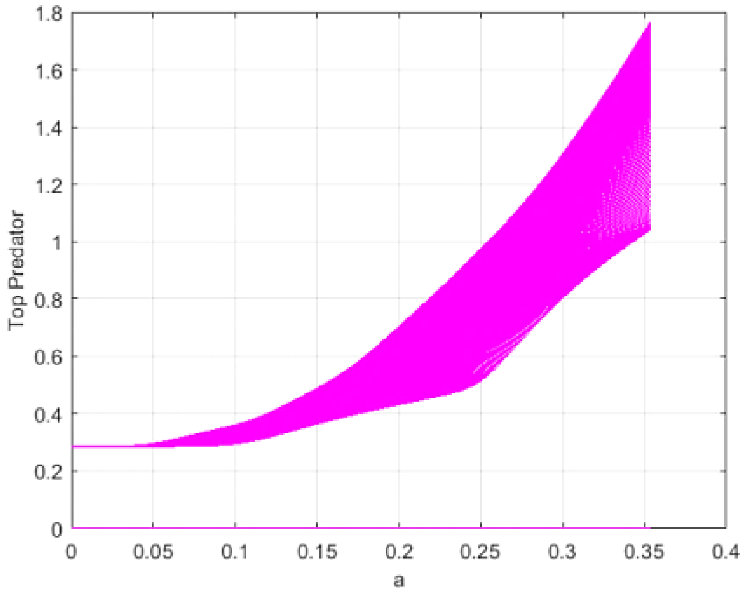
(a)


Medium Predator-1
a
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0
1
2
3
4
5
6
7

(b)

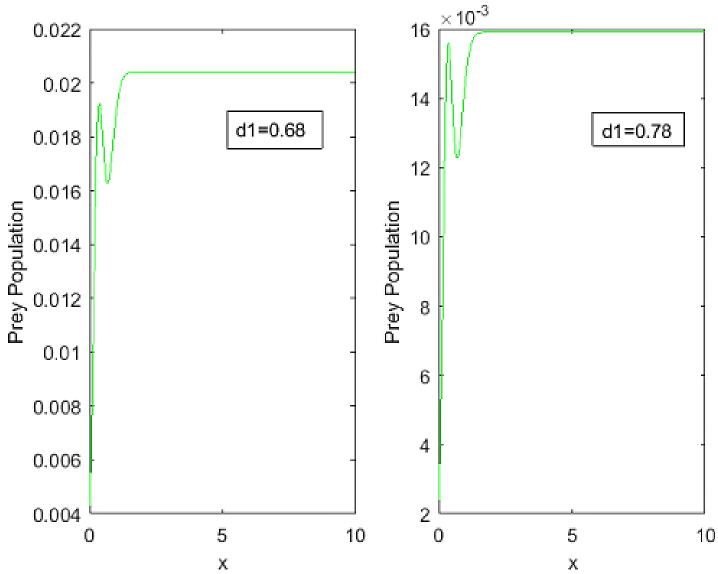

Medium Predator-2
a
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0
1
2
3
4
5
6

(c)

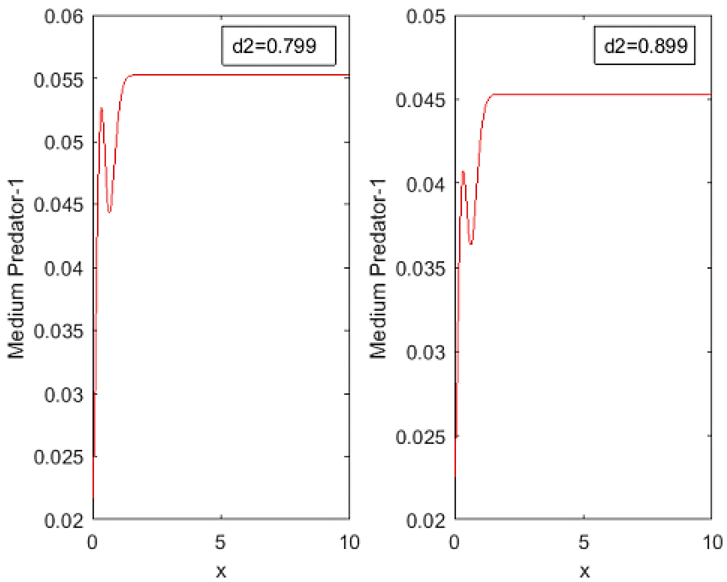


(d)

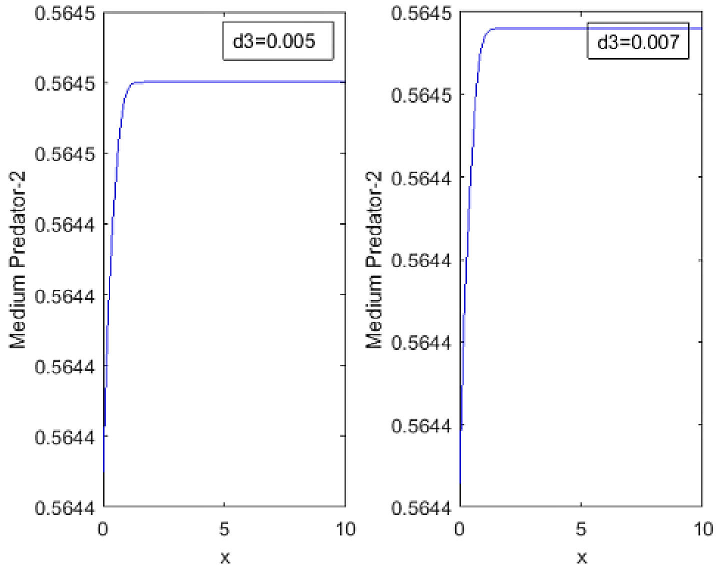
Figure 13. **Bifurcation diagram for the non-diffusive model**



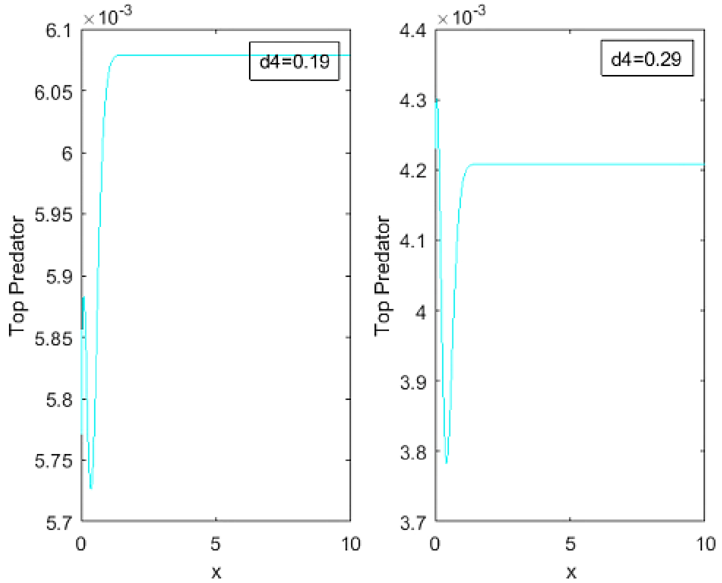
(a)



(b)



(c)



(d)

Figure 14. Impact of diffusion coefficients on the interactive population

The present work focuses on the global existence and stability of equilibrium states in a four-species food chain model, utilizing Lyapunov functionals and bifurcation analysis, particularly focusing on the role of prey growth rate as a bifurcation parameter. Additionally, artificial neural networks (ANN) are incorporated to explore complex dynamics. The study by Jin et al. [40] emphasizes global existence using coupling energy estimates in two-dimensional spaces. It applies Lyapunov functionals along with LaSalle's invariance principle to establish the stability of prey-only, semi-coexistence, and coexistence steady states. At the same time, the present study highlights diffusive and non-diffusive dynamics. Both studies contribute to understanding equilibrium stability, but with different methodologies, we explore bifurcation and modern computational tools, whereas Jin et al. [40] focused on coupling energy techniques and invariance principles.

7- Conclusion

In this study, a comprehensive investigation into a food chain model involving four species is conducted, considering both diffusive and non-diffusive versions of the model. By establishing criteria for the existence and stability of equilibrium points within a bounded region in terms of system parameters, the study provided valuable analysis of the dynamics of the system. Utilizing analysis techniques, positivity and boundedness are discussed. The equilibrium points are identified, and conditions for their existence are outlined. Stability analysis is performed using the Lyapunov function, and a numerical scheme based on three-time levels and the system's behavior under various conditions is discussed. The results showed that diffusion has a major effect on the dynamics of interacting species' populations, highlighting how important it is to include diffusion in the model. In particular, bifurcation plots highlighted the significance of prey growth rate as a bifurcation parameter, providing greater insight into the system's behavior. Decoding the complex food chain model using ANN gives a better insight into species interaction, dynamics, and behaviors. The article concludes with a call for more significant research into more complicated dynamics by recommending that this line of inquiry be expanded to investigate stability and bifurcation in the context of a fractional-order food chain model.

The research also overlays the way modern computational techniques, such as hybrid modeling techniques and data-driven methods, are integrated to study ecological systems. By bridging traditional analytical methods with emerging technologies, future studies could unveil concealed patterns in population interactions and responses to environmental changes. Expanding the study to add stochastic effects, environmental variability, and multi-scale interactions would supplement the understanding of real-world dynamics. Moreover, investigating anthropogenic influences, such as pollution effects and habitat fragmentation, within the fractional-order framework could offer valuable insights into managing ecosystems under stress. Such modifications would not only deepen the theoretical understanding of ecological models but also contribute to practical strategies for biodiversity conservation and sustainable ecosystem management.

8- Declarations

8-1- Author Contributions

Conceptualization, M.S.A. and A.E.; methodology, A.E.; software, A.E.; validation, M.S.A., A.E., and A.U.R.; formal analysis, A.U.R.; investigation, M.S.A.; resources, M.S.A.; data curation, A.E.; writing—original draft preparation, M.S.A.; writing—review and editing, A.U.R.; visualization, A.E.; supervision, M.S.A.; project administration, M.S.A.; funding acquisition, A.U.R. All authors have read and agreed to the published version of the manuscript.

8-2- Data Availability Statement

The data presented in this study are available on request from the corresponding author.

8-3- Funding and Acknowledgments

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8-4- Institutional Review Board Statement

Not applicable.

8-5- Informed Consent Statement

Not applicable.

8-6- Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this manuscript. In addition, the ethical issues, including plagiarism, informed consent, misconduct, data fabrication and/or falsification, double publication and/or submission, and redundancies have been completely observed by the authors.

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