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LH-Moments Parameter Estimation of Weibull Distribution

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Abstract

Natural disasters such as sudden floods, storms, severe snowfall, and droughts are major problems in the world. Generally the distributions of extreme values are heavy-tailed distributions, and an important heavy-tailed distribution is the Weibull distribution, especially for non-linear behaviors. Therefore, accurately estimation of the occurrence of disasters is required to deal with such situations in a timely and efficient manner. Several methods can be used to estimate the parameters, for example, moments estimate, maximum likelihood estimate, linear of moment, and high-order L-moments. The objectives of this article are to estimate the parameters of the four-parameter Weibull distribution with weak non-linear effects (W4DN) based on the LH-moments method, and to propose a new parameter estimation formula. The proposed formula is classified into two cases based on the coefficient of the second-order term (δ): Case 1, where the coefficient is positive ($\delta > 0$) and Case 2, where the coefficient is negative ($\delta < 0$). In both cases, the corresponding estimation formulas are derived β_r and λ_r^p for $p=1, 2, \ldots$ and $r=1, 2, \ldots$, respectively. The parameter estimations ($\hat{\gamma}, \hat{\alpha}, \hat{\delta}, \hat{\phi}$ and $\hat{\kappa}$) are then optimized using the augmented Lagrangian adaptive barrier minimization algorithm. These formulas provide a practical approach for parameter estimation that is essential for forecasting extreme events in various disciplines, including hydrology, meteorology, insurance, finance, and engineering.

Keywords:

Weibull Distribution;
Non-Linear Effects;
L-Moments;
LH-Moments;
Extreme Event.

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1- Introduction

Nowadays, the data used for predicting extreme events, such as extreme rainfall, extreme temperatures, flood, drought and height of wave, in order to applied in many fields for meteorology, hydrology, agriculture and engineering. Mostly, the distributions of extreme value are generalized extreme value distribution, generalized Pareto distribution, Kappa distribution, Wakeby distribution and Weibull distribution. Many researchers have studied and developed these distributions, including the Weibull distribution. The Weibull distribution is commonly used for modeling phenomena with monotonic failure rates [1, 2] and is used to model lifetime data. A variety of new distributions were developed as generalizations of the Weibull distribution. For instance, Mudholkar & Srivastava [3] modified the exponential Weibull distribution. Xie & Lai [4] added two Weibull survival functions into the Weibull model. Xie et al. [5] proposed a Weibull extension distribution. Lai et al. [6] developed a modified Weibull distribution. Bebbington et al. [7] studied a flexible Weibull distribution and its properties. Lai [8] reviewed some generalized Weibull distributions. Almheidat et al. [9] proposed a generalization of the Weibull distribution, namely, the Lomax-Weibull distribution, that describes the shape and scale parameter properties. Meanwhile, Kamal & Ismail [10] suggested a new five parameter lifetime distribution, which mixed the complex Weibull extension distribution and the Burr XII distribution. A type of interesting Weibull distributions involves nonlinear behavior data to forecast the extreme characteristic for the estimation parameter

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process which should accurately reflect the tail of the distribution. Nonlinear behavior data has been previously studied by Izadparast & Niedzwecki [11], which involved a four-parameter Weibull distribution with weak non-linear effects. They investigated the probability distribution function, commonly observed as wave data for engineering applications in offshore environments, which requires an accurate forecasting model. The behavior of the tail distribution is the most important to study.

In addition to studying the probability distribution of data behavior, determination of the parameters for practical applications is essential. Accurate and reliable parameter estimation is crucial, so the estimation of distribution parameters has attracted considerable interest. Several methods have been proposed for parameter estimation, including the method of moments (MM), maximum likelihood estimation (MLE), L-moments, and higher-order L-moments (LH-moments). The method of moments, introduced by Pafnuty Chebyshev [12], and the maximum likelihood estimation method, developed by R.A. Fisher between 1912 and 1922 [13], are among the foundational techniques in parameter estimation. More recently, researchers have applied Artificial Neural Networks (ANNs) as an alternative approach for parameter estimation, such as Phoophiwfa et al. [14].

Hosking [15] introduced a new approach known as L-moments, which is based on linear combinations of order statistics and serve as an alternative to conventional product moments for describing probability distributions and sample data. Compared to traditional moments, L-moments are more robust in relation to extreme values and tail behavior and provide more accurate parameter estimates, especially when dealing with small sample sizes. Furthermore, when compared with maximum likelihood estimation, L-moments often yield more efficient and less biased parameter estimates in small-sample contexts. Several researchers have applied the L-moments method in various studies. Notably, Busababodhin et al. [16], Prahadchai et al. [17], and Shin et al. [18] have applied this technique with real data. Additionally, comparisons between L-moment-based estimates and other methods have been investigated in various studies, such as Khan et al. [19, 20] and Bakar et al. [21].

Even though the L-moments method is quite useful to estimate parameters, this method may not be suitable for cases that involve predicting long return period situations. Moreover, the L-moments method is oversensitive to the lower part of tail of the distribution and provides insufficient weight to large data values that actually provide valuable information on the upper tail of the distribution. Therefore, Wang [22] improved L-moments to be high-order L-moments (LH-moments), so L- moments are a special case of LH-moments, which focus on the upper part of the tail distribution as well as extreme events more precisely. The estimation method of LH-moments normally indicates high upper quantiles rather than lower quantiles of the distribution. Some researchers have applied LH-moments in the area of extreme events, such as hydrology, meteorology, structural engineering, irrigation engineering, finance, and insurance. These studies applied four-parameter Kappa distribution (K4D), generalized extreme value distribution (GEV), generalized Pareto distribution (GPD), generalized logistic distribution (GLD), and Wakeby distribution (WAD), such as Wang [22], Meshgi & Khalili [23], Murshed et al. [24], Busababodhin et al. [25] and Piyapatr et al. [26], Anghel & Ilinca [27], and Singh & Chavan [28].

As discussed above, parameter estimation using LH-moments has been shown to be more suitable than L-moments for modeling data with upper tails. Therefore, in this paper, new parameter estimation formulas based on LH-moments for the four-parameter Weibull distribution with weak non-linear effects (W4DN) are proposed and presented. The remainder of this paper is organized as follows: Weibull distribution with four parameters and weak non-linear effects and definition of LH-moments are explained in Section 2 and Section 3, respectively. The formulas of parameter estimation for W4DN using LH-moments method are derived in Section 4. Finally, the study conclusion is presented in Section 5.

2- Weibull Distribution with Four Parameters and Weak Non-Linear Effects

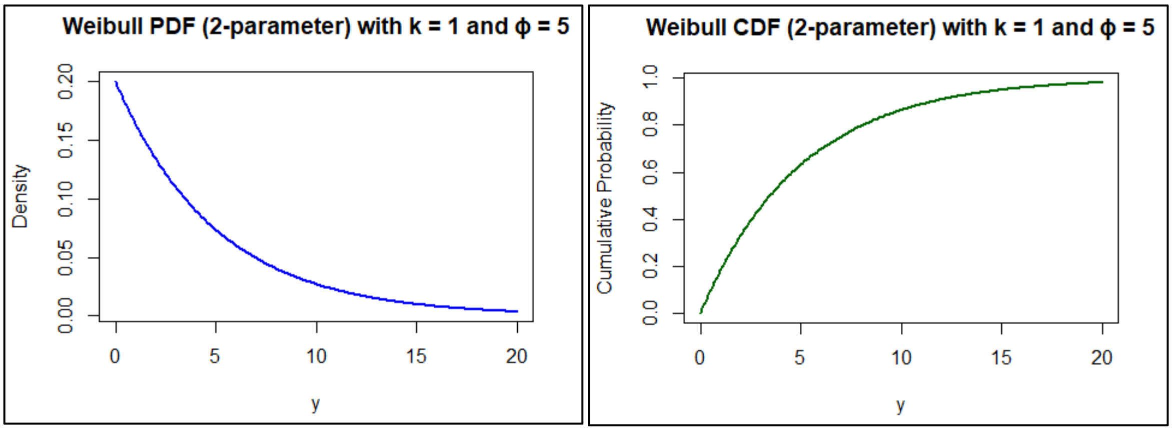
Introduced by Waloddi Weibull in 1951, the Weibull distribution is continuous [29]. The two-parameter Weibull distribution has probability density function (PDF) and cumulative distribution function (CDF) which are defined as follows:

$$f(y) = \frac{\kappa}{\phi}\left(\frac{y}{\phi}\right)^{\kappa-1} exp\left(-\left(\frac{y}{\phi}\right)^{\kappa}\right); y > 0, \tag{1}$$

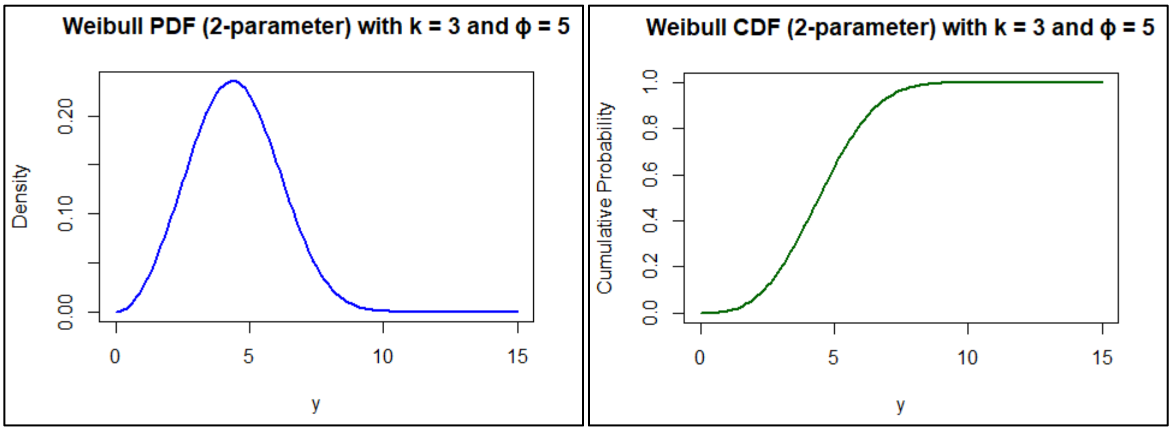
and

$$F(y) = 1 - exp\left(-\left(\frac{y}{\phi}\right)^{\kappa}\right); y > 0, \tag{2}$$

where y is a random variable, ϕ is the scale parameter, and κ is the shape parameter. The PDF and the CDF of a Weibull distribution with two parameters is shown in Figure 1.



(a)



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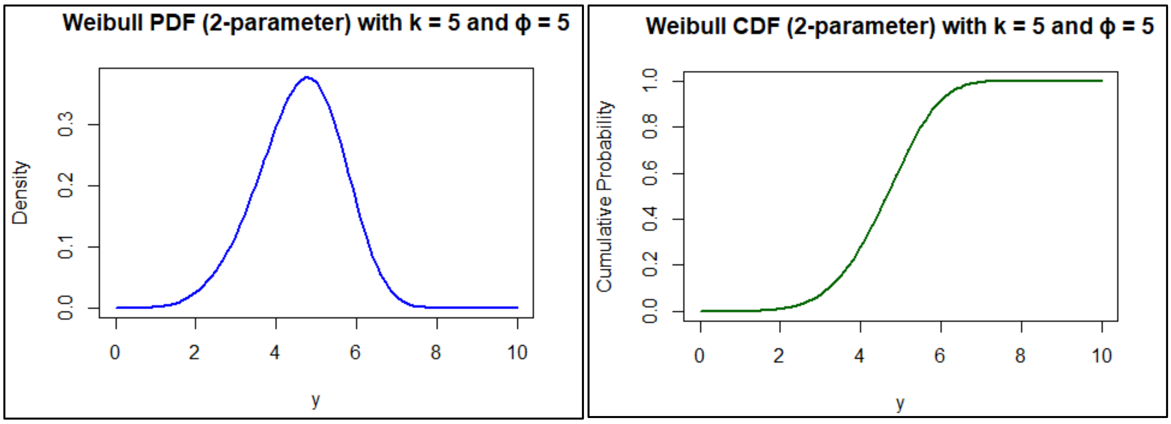

(c)

Figure 1. The PDF and CDF of a Weibull distribution with two parameters for varying shape and scale = 5

The Weibull distribution with four parameters and weak non-linear effects (W4DN) is focused on second-order Stokes wave theory, following Izadparast & Niedzwecki [11], in which the peaks and valleys ζ_n of weakly non-linear and narrow banded process can be written to closed form in the quadratic relationship with the linear random variable ζ, defined as $\zeta_n = \gamma + \alpha\zeta + \delta\zeta^2$, where γ is the constant linear shift (zeroth-order term) between the linear and non-linear process and α and δ are the coefficient of the first-order and second-order terms, respectively. The conditions of parameters, γ, α, and δ are real numbers, $\alpha \geq 0$ and $|\delta| \ll \alpha$.

The Weibull distribution with four-parameter has probability density function, cumulative distribution function, and quantile function (f_{ζ_n} , F_{ζ_n}, and y_{ζ_n}respectively) of the non-linear random variables ζ_n, which is defined as follows:

In case of $\delta > 0$,

$$f_{\zeta_n}(y) = \frac{\kappa}{\phi y}\left(\frac{y-\alpha}{2\delta\phi}\right)^{\kappa-1} exp\left(-\left(\frac{-y-\alpha}{2\delta\phi}\right)^{\kappa}\right); y > \gamma, \tag{3}$$

$$F_{\zeta_n}(y) = 1 - exp\left(-\left(\frac{y-\alpha}{2\delta\phi}\right)^{\kappa}\right), \tag{4}$$

and;

$$y_{\zeta_n}(u) = \gamma + \delta\phi^2[-ln(1-u)]^{\frac{1}{\kappa}} + \alpha\phi[-ln(1-u)]^{\frac{1}{\kappa}}, \tag{5}$$

where; $0 \le u < 1$ and $u = P(\zeta_n \le y_\zeta)$.

In case of $\delta < 0$,

$$f_{\zeta_n}(y) = \frac{\kappa}{\phi y}\left[\left(\frac{y-\alpha}{2\delta\phi}\right)^{\kappa-1} exp\left(-\left(\frac{-y-\alpha}{2\delta\phi}\right)^{\kappa}\right) H_\gamma(y) + \left(\frac{-y-\alpha}{2\delta\phi}\right)^{\kappa-1} exp\left(-\left(\frac{-y-\alpha}{2\delta\phi}\right)^{\kappa}\right)\right], \tag{6}$$

where; $y \le \gamma - \frac{\alpha^2}{4\delta}$,

$$F_{\zeta_n}(y) = \left[1 - exp\left(-\left(\frac{y-\alpha}{2\delta\phi}\right)^{\kappa}\right)\right] H_\gamma(y) + exp\left(-\left(\frac{-y-\alpha}{2\delta\phi}\right)^{\kappa}\right), \tag{7}$$

where; $H_\gamma(y) = \begin{cases} 1; y \ge \gamma \\ 0; y < \gamma \end{cases}$,

and

$$y_{\zeta_n}(u) \approx \gamma + \delta\phi^2[-ln(1-u)]^{\frac{2}{\kappa}} + \alpha\phi[-ln(1-u)]^{\frac{1}{\kappa}}, \tag{8}$$

where; $0 \le u \le 1 - exp\left(-\left(\frac{-\alpha}{2\delta\phi}\right)^{\kappa}\right)$.

3- Definition of LH-moments

3-1- Extreme Value Theory

Wang (1997) [22] defined the LH-moments which are a linear combination of high-order statistics and the formula of r-th LH-moments with order $p = 0, 1, 2, \ldots$ for $r = 1, 2, \ldots$ as $\lambda_r^p = \sum_{k=0}^{r-1} C_{r,k} E[Y_{p+r-k:p+r}]$ where, $C_{r,k} = (-1)^k \binom{r-1}{k}$, if $p = 0$ then the LH-moment is the same as the L-moment.

The first five LH-moments are follows:

$\lambda_1^p = E[Y_{p+1:p+1}]$ is the measure of the location of the distribution,

$\lambda_2^p = \frac{1}{2}E[Y_{p+2:p+2} - Y_{p+1:p+2}]$ is the spreadness of the upper part of tail distribution,

$\lambda_3^p = \frac{1}{3}E[Y_{p+3:p+3} - 2Y_{p+2:p+3} + Y_{p+1:p+3}]$ shows the asymmetric of the upper part of tail distribution,

$\lambda_4^p = \frac{1}{4}E[Y_{p+4:p+4} - 3Y_{p+3:p+4} + 3Y_{p+2:p+4} - Y_{p+1:p+4}]$ shows the peak of the upper part of tail distribution,

$\lambda_5^p = \frac{1}{5}E[Y_{p+5:p+5} - 4Y_{p+4:p+5} + 6Y_{p+3:p+5} - 4Y_{p+2:p+5} + Y_{p+1:p+5}]$

The LH-moments of coefficient of variation, LH-moments of skewness, and LH-moments of kurtosis are defined as

LH-moments of coefficient of variation: $\tau_2^p = \frac{\lambda_2^p}{\lambda_1^p}$, (9)

LH-moments of skewness: $\tau_3^p = \frac{\lambda_3^p}{\lambda_2^p}$ (10)

LH-moments of kurtosis: $\tau_4^p = \frac{\lambda_4^p}{\lambda_2^p}$. (11)

Accordingly, the LH-moments can be written in terms of probability weighted moments (PWMs) by Greenwood et al. [30] for $p = 0, 1, 2, \ldots$ and $r = 1, 2, \ldots$ as

$$\lambda_r^p = \frac{p+r}{r!}\left[\sum_{k=0}^{r-1} C_{r,k}\left(\prod_{j=1}^{r-2}(p+r+j-k)\right)B_{p+r-k-1}\right], \tag{12}$$

where normalized PWMs is defined in the form:

$$B_r = (r+1)\beta_r = (r+1)\int_0^1 y(F)F^r dF, \tag{13}$$

when β_r is the standard PWMs.

Wang (1997) [22] derived the LH-moments of the distribution from the relationship between normalized PWMs and LH-moments as follows:

$$\lambda_1^p = B_p, \tag{14}$$

$$\lambda_2^p = \frac{p+2}{2!}\left[B_{p+1} - B_p\right], \tag{15}$$

$$\lambda_3^p = \frac{p+3}{3!}\left[(p+4)B_{p+2} - 2(p+3)B_{p+1} + (p+2)B_p\right], \tag{16}$$

$$\lambda_4^p = \frac{p+4}{4!}\left[(p+6)(p+5)B_{p+3} - 3(p+5)(p+4)B_{p+2} + 3(p+4)(p+3)B_{p+1} - (p+3)(p+2)B_p\right], \tag{17}$$

$$\lambda_5^p = \frac{p+5}{5!}\begin{bmatrix}(p+8)(p+7)(p+6)B_{p+4} - 4(p+7)(p+6)(p+5)B_{p+3} + 6(p+6)(p+5)(p+4)B_{p+2} \\ -4(p+5)(p+4)(p+3)B_{p+1} + (p+4)(p+3)(p+2)B_p\end{bmatrix} \tag{18}$$

The relation of Equations 13 to 18 are used to derive LH-moments of the W4DN. The steps taken in this research are depicted in Figure 2.

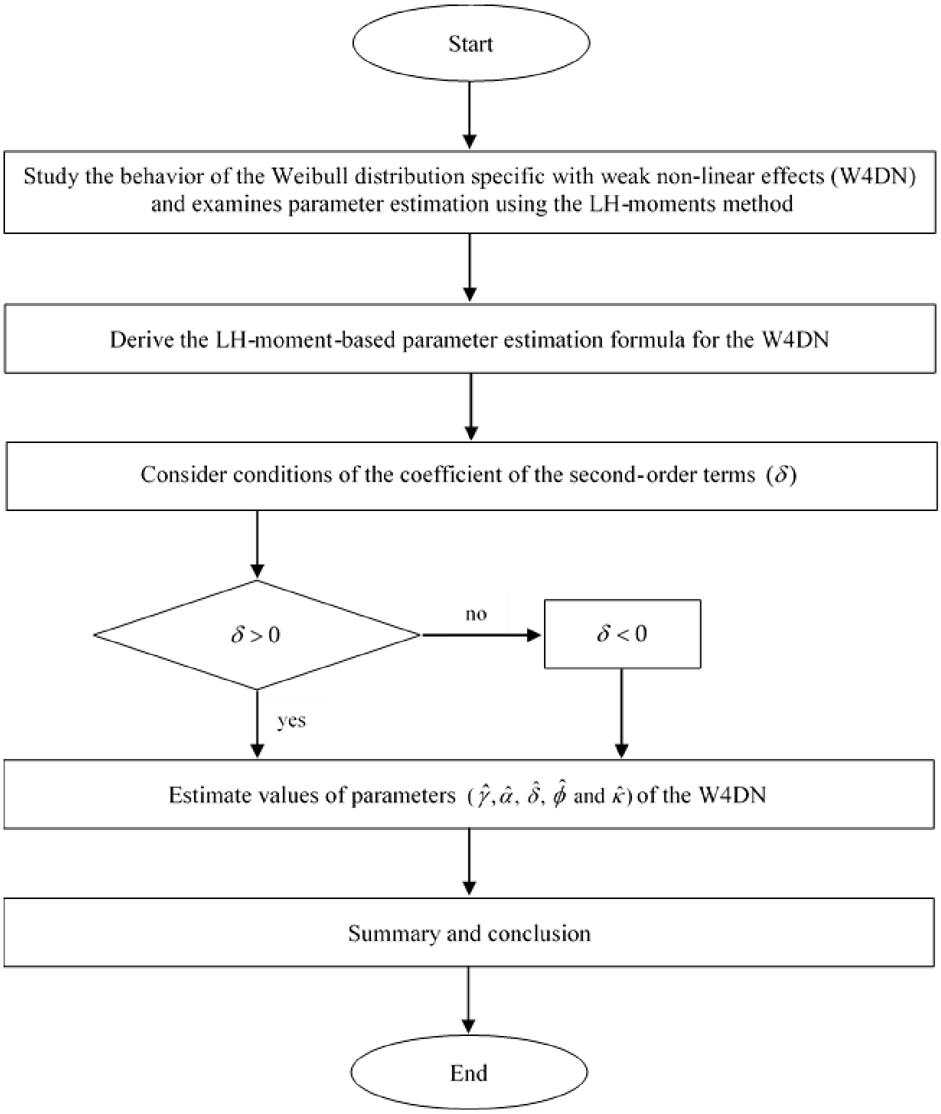

Figure 2. Overview of the Research Methodology Procedures

The formulas of LH-moments of the W4DN are shown in the next section.

4- Result

4-1- LH-moments of W4DN

The LH-moments method for parameter estimation of the W4DN focuses on the quadratic parameter in the form $\zeta_n = \gamma + \alpha\zeta + \delta\zeta^2$, where γ is the constant linear shift (zeroth-order term), and α and δ are the coefficients of the first-order and second-order terms, respectively. When the second-order coefficient is equal to zero ($\delta = 0$), the W4DN reduces to the W4D. Therefore, the parameter estimation procedure is divided into two cases based on the coefficient of the second-order term (δ), case 1: $\delta > 0$ (positive second-order coefficient) and case 2: $\delta < 0$ (negative second-order coefficient). The derivation of the parameter estimation formulas for each case is outlined as follows.

In case of $\delta > 0$, when $y(F) = \gamma + \delta\phi^2[-ln(1-F)]^{\frac{1}{\kappa}} + \alpha\phi[-ln(1-F)]^{\frac{1}{\kappa}}$ and $\beta_r = \int_0^1 x(F)F^r dF$

We get,

$$\beta_r = \frac{1}{r+1}\gamma + (\delta\phi^2 + \alpha\phi)\int_0^1 [-ln(1-F)]^{\frac{1}{\kappa}} F^r dF, \tag{19}$$

Consider, $\int_0^1 [-ln(1-F)]^{\frac{1}{\kappa}} F^r dF$ and let $u = -ln(1-F)$, therefore,

$$\int_0^1 [-ln(1-F)]^{\frac{1}{\kappa}} F^r dF = \int_0^\infty u^{\frac{1}{\kappa}} (1-e^{-u})^r e^{-u} du$$

$$= \int_0^\infty u^{\frac{1}{\kappa}} e^{-u} [1 + (-e^{-u})]^r du$$

$$= \int_0^\infty u^{\frac{1}{\kappa}} e^{-u} \sum_{j=0}^r (-1)^j \binom{r}{j} (e^{-uj})\, du, \text{ from Binomial theory}$$

$$= \sum_{j=0}^r (-1)^j \binom{r}{j} \int_0^\infty u^{\frac{1}{\kappa}} e^{-u(j+1)} du$$

$$= \sum_{j=0}^r (-1)^j \binom{r}{j} \left(\frac{1}{j+1}\right)^{\frac{1}{\kappa}+1} \int_0^\infty [(j+1)u]^{\left(\frac{1}{\kappa}+1\right)-1} e^{-(j+1)u} d(j+1)u$$

$$= \sum_{j=0}^r (-1)^j \binom{r}{j} \left(\frac{1}{j+1}\right)^{\frac{1}{\kappa}+1} \Gamma\left(\frac{1}{\kappa}+1\right), \text{ from Gamma function} \tag{20}$$

substituted Equation 20 intoEquation 19, then

$$\beta_r = \frac{1}{r+1}\gamma + (\delta\phi^2 + \alpha\phi)\sum_{j=0}^r (-1)^j \binom{r}{j} \left(\frac{1}{j+1}\right)^{\frac{1}{\kappa}+1} \Gamma\left(\frac{1}{\kappa}+1\right)$$

$$= \frac{1}{r+1}\gamma + (\delta\phi^2 + \alpha\phi)\Gamma\left(\frac{1}{\kappa}+1\right)\sum_{j=0}^r (-1)^j \binom{r}{j} \left(\frac{1}{j+1}\right)^{\frac{1}{\kappa}+1}$$

$$= \frac{1}{r+1}\gamma + (\delta\phi^2 + \alpha\phi)\Gamma\left(\frac{1}{\kappa}+1\right)\sum_{j=0}^r (-1)^j \binom{r}{j} (j+1)^{-\frac{1}{\kappa}} \left(\frac{1}{j+1}\right)$$

$$\therefore (r+1)\beta_r = \gamma + (r+1)(\delta\phi^2 + \alpha\phi)\Gamma\left(\frac{1}{\kappa}+1\right)\sum_{j=0}^r (-1)^j (j+1)^{-\frac{1}{\kappa}} \binom{r}{j} \left(\frac{1}{j+1}\right)$$

$$= \gamma + \frac{(\delta\phi^2 + \alpha\phi)}{\kappa}\Gamma\left(\frac{1}{\kappa}\right)\sum_{j=0}^r (-1)^j (j+1)^{-\frac{1}{\kappa}} \binom{r+1}{j+1}.$$

So,

$$B_r = \gamma + \frac{(\delta\phi^2 + \alpha\phi)}{\kappa}\Gamma\left(\frac{1}{\kappa}\right)\sum_{j=0}^r (-1)^j (j+1)^{-\frac{1}{\kappa}} \binom{r+1}{j+1}$$

when $r = p$; $\quad B_p = \gamma + \frac{(\delta\phi^2 + \alpha\phi)}{\kappa}\Gamma\left(\frac{1}{\kappa}\right)\sum_{j=0}^p (-1)^j (j+1)^{-\frac{1}{\kappa}} \binom{p+1}{j+1}$, (21)

when $r = p+1$; $B_{p+1} = \gamma + \frac{(\delta\phi^2 + \alpha\phi)}{\kappa}\Gamma\left(\frac{1}{\kappa}\right)\sum_{j=0}^{p+1} (-1)^j (j+1)^{-\frac{1}{\kappa}} \binom{p+2}{j+1}$, (22)

when $r = p+2$; $B_{p+2} = \gamma + \frac{(\delta\phi^2 + \alpha\phi)}{\kappa}\Gamma\left(\frac{1}{\kappa}\right)\sum_{j=0}^{p+2} (-1)^j (j+1)^{-\frac{1}{\kappa}} \binom{p+3}{j+1}$, (23)

when $r = p+3$; $B_{p+3} = \gamma + \frac{(\delta\phi^2 + \alpha\phi)}{\kappa}\Gamma\left(\frac{1}{\kappa}\right)\sum_{j=0}^{p+3} (-1)^j (j+1)^{-\frac{1}{\kappa}} \binom{p+4}{j+1}$, (24)

when $r = p+4$; $B_{p+4} = \gamma + \frac{(\delta\phi^2 + \alpha\phi)}{\kappa}\Gamma\left(\frac{1}{\kappa}\right)\sum_{j=0}^{p+4} (-1)^j (j+1)^{-\frac{1}{\kappa}} \binom{p+5}{j+1}$, (25)

Equations 21 to 25 serve as the initial expressions for computing the first five LH-moments ($\lambda_1^p, \lambda_2^p, \lambda_3^p, \lambda_4^p$ and λ_5^p). These moments are obtained by substituting Equations 21 to 25 into Equations 14 to 18, respectively. The resulting expressions are as follows:

$$\lambda_1^p = \gamma + \frac{(\delta\phi^2 + \alpha\phi)}{\kappa}\Gamma\left(\frac{1}{\kappa}\right)\sum_{j=0}^p (-1)^j (j+1)^{-\frac{1}{\kappa}} \binom{p+1}{j+1}, \tag{26}$$

$$\lambda_2^p = \frac{p+2}{2!}\frac{(\delta\phi^2 + \alpha\phi)}{\kappa}\Gamma\left(\frac{1}{\kappa}\right)\sum_{j=0}^{p+1} (-1)^j (j+1)^{-\frac{1}{\kappa}} \binom{p+1}{j}, \tag{27}$$

$$\lambda_3^p = \frac{p+3}{3!}\frac{(\delta\phi^2 + \alpha\phi)}{\kappa}\Gamma\left(\frac{1}{\kappa}\right)\sum_{j=0}^{p+2} (-1)^j (j+1)^{-\frac{1}{\kappa}} (j+2) \binom{p+2}{j}, \tag{28}$$

$$\lambda_4^p = \frac{p+4}{4!}\frac{(\delta\phi^2+\alpha\phi)}{\kappa}\Gamma\left(\frac{1}{\kappa}\right)\sum_{j=0}^{p+3}(-1)^j(j+1)^{-\frac{1}{\kappa}}(j+2)(j+3)\binom{p+3}{j}, \tag{29}$$

$$\lambda_5^p = \frac{p+5}{5!}\frac{(\delta\phi^2+\alpha\phi)}{\kappa}\Gamma\left(\frac{1}{\kappa}\right)\sum_{j=0}^{p+4}(-1)^j(j+1)^{-\frac{1}{\kappa}}(j+2)(j+3)(j+4)\binom{p+4}{j}. \tag{30}$$

Similarly, in the case of $\delta < 0$, when $y(F) \approx \gamma + \delta\phi^2[-\ln(1-F)]^{\frac{2}{\kappa}} + \alpha\phi[-\ln(1-F)]^{\frac{1}{\kappa}}$ and $\beta_r = \int_0^1 x(F)F^r dF$.

$$\begin{aligned}\beta_r &= \int_0^1 \left[\gamma + \delta\phi^2[-\ln(1-F)]^{\frac{2}{\kappa}} + \alpha\phi[-\ln(1-F)]^{\frac{1}{\kappa}}\right] F^r dF \\ &= \gamma\int_0^1 F^r dF + \delta\phi^2\int_0^1[-\ln(1-F)]^{\frac{2}{\kappa}}F^r dF + \alpha\phi\int_0^1[-\ln(1-F)]^{\frac{1}{\kappa}}F^r dF.\end{aligned} \tag{31}$$

Consider, the first term of Equation 31, then

$$\gamma\int_0^1 F^r dF = \frac{1}{r+1}\gamma. \tag{32}$$

Consider, the second term of Equation 31 and let $u = -\ln(1-F)$, therefore,

$$\int_0^1[-\ln(1-F)]^{\frac{2}{\kappa}} F^r dF = \int_0^\infty u^{\frac{2}{\kappa}}(1-e^{-u})^r e^{-u} du$$

$= \int_0^\infty u^{\frac{2}{\kappa}} e^{-u}\sum_{j=0}^r(-1)^j\binom{r}{j}(e^{-uj})\, du$, from Binomial theorem.

So, $\delta\phi^2\int_0^1[-\ln(1-F)]^{\frac{2}{\kappa}}F^r dF = \delta\phi^2\int_0^\infty u^{\frac{2}{\kappa}} e^{-u}\sum_{j=0}^r(-1)^j\binom{r}{j}(e^{-uj})\, du$

$$= \delta\phi^2\sum_{j=0}^r(-1)^j\binom{r}{j}\left(\frac{1}{j+1}\right)^{\frac{2}{\kappa}+1}\int_0^\infty[(j+1)u]^{\left(\frac{2}{\kappa}+1\right)-1} e^{-[(j+1)u]}d[(j+1)u]$$

$= \delta\phi^2\sum_{j=0}^r(-1)^j\binom{r}{j}\left(\frac{1}{j+1}\right)^{\frac{2}{\kappa}+1}\Gamma\left(\frac{2}{\kappa}+1\right)$, from Gamma function

$$= \frac{1}{r+1}\delta\phi^2\frac{2}{\kappa}\Gamma\left(\frac{2}{\kappa}\right)\sum_{j=0}^r(-1)^j(j+1)^{-\frac{2}{\kappa}}\binom{r+1}{j+1} \tag{33}$$

Consider, the last term of Equation 31, in the same way of the second term, then

$$\int_0^1[-\ln(1-F)]^{\frac{1}{\kappa}} F^r dF = \int_0^\infty u^{\frac{1}{\kappa}} e^{-u}\sum_{j=0}^r(-1)^j\binom{r}{j}(e^{-uj})\, du.$$

So, $\alpha\phi\int_0^1[-\ln(1-F)]^{\frac{1}{\kappa}}F^r dF = \alpha\phi\int_0^\infty u^{\frac{1}{\kappa}} e^{-u}\sum_{j=0}^r(-1)^j\binom{r}{j}(e^{-uj})\, du$

$$= \alpha\phi\sum_{j=0}^r(-1)^j\binom{r}{j}\left(\frac{1}{j+1}\right)^{\frac{1}{\kappa}+1}\int_0^\infty[(j+1)u]^{\left(\frac{1}{\kappa}+1\right)-1} e^{-[(j+1)u]}d[(j+1)u]$$

$= \alpha\phi\sum_{j=0}^r(-1)^j\binom{r}{j}\left(\frac{1}{j+1}\right)^{\frac{1}{\kappa}+1}\Gamma\left(\frac{1}{\kappa}+1\right)$, from Gamma function

$$= \frac{1}{r+1}\alpha\phi\frac{1}{\kappa}\Gamma\left(\frac{1}{\kappa}\right)\sum_{j=0}^r(-1)^j(j+1)^{-\frac{1}{\kappa}}\binom{r+1}{j+1} \tag{34}$$

substituted Equations 32 to 34 into Equation 31, then

$$\beta_r = \frac{1}{r+1}\gamma + \frac{1}{r+1}\delta\phi^2\frac{2}{\kappa}\Gamma\left(\frac{2}{\kappa}\right)\sum_{j=0}^r(-1)^j(j+1)^{-\frac{2}{\kappa}}\binom{r+1}{j+1} + \frac{1}{r+1}\alpha\phi\frac{1}{\kappa}\Gamma\left(\frac{1}{\kappa}\right)\sum_{j=0}^r(-1)^j(j+1)^{-\frac{1}{\kappa}}\binom{r+1}{j+1}.$$

So,

$$B_r = \gamma + \frac{2\delta\phi^2}{\kappa}\Gamma\left(\frac{2}{\kappa}\right)\sum_{j=0}^r(-1)^j(j+1)^{-\frac{2}{\kappa}}\binom{r+1}{j+1} + \frac{\alpha\phi}{\kappa}\Gamma\left(\frac{1}{\kappa}\right)\sum_{j=0}^r(-1)^j(j+1)^{-\frac{1}{\kappa}}\binom{r+1}{j+1},$$

when; $r = p$;

$$B_p = \gamma + \frac{2\delta\phi^2}{\kappa}\Gamma\left(\frac{2}{\kappa}\right)\sum_{j=0}^p(-1)^j(j+1)^{-\frac{2}{\kappa}}\binom{p+1}{j+1} + \frac{\alpha\phi}{\kappa}\Gamma\left(\frac{1}{\kappa}\right)\sum_{j=0}^p(-1)^j(j+1)^{-\frac{1}{\kappa}}\binom{p+1}{j+1}, \tag{35}$$

when; $r = p+1$;

$$B_{p+1} = \gamma + \frac{2\delta\phi^2}{\kappa}\Gamma\left(\frac{2}{\kappa}\right)\sum_{j=0}^{p+1}(-1)^j(j+1)^{-\frac{2}{\kappa}}\binom{p+2}{j+1} + \frac{\alpha\phi}{\kappa}\Gamma\left(\frac{1}{\kappa}\right)\sum_{j=0}^{p+1}(-1)^j(j+1)^{-\frac{1}{\kappa}}\binom{p+2}{j+1}, \tag{36}$$

when $r = p + 2$;

$$B_{p+2} = \gamma + \frac{2\delta\phi^2}{\kappa}\Gamma\left(\frac{2}{\kappa}\right)\sum_{j=0}^{p+2}(-1)^j(j+1)^{-\frac{2}{\kappa}}\binom{p+3}{j+1} + \frac{\alpha\phi}{\kappa}\Gamma\left(\frac{1}{\kappa}\right)\sum_{j=0}^{p+2}(-1)^j(j+1)^{-\frac{1}{\kappa}}\binom{p+3}{j+1}, \tag{37}$$

when $r = p + 3$;

$$B_{p+3} = \gamma + \frac{2\delta\phi^2}{\kappa}\Gamma\left(\frac{2}{\kappa}\right)\sum_{j=0}^{p+3}(-1)^j(j+1)^{-\frac{2}{\kappa}}\binom{p+4}{j+1} + \frac{\alpha\phi}{\kappa}\Gamma\left(\frac{1}{\kappa}\right)\sum_{j=0}^{p+3}(-1)^j(j+1)^{-\frac{1}{\kappa}}\binom{p+4}{j+1}, \tag{38}$$

when $r = p + 4$;

$$B_{p+4} = \gamma + \frac{2\delta\phi^2}{\kappa}\Gamma\left(\frac{2}{\kappa}\right)\sum_{j=0}^{p+4}(-1)^j(j+1)^{-\frac{2}{\kappa}}\binom{p+5}{j+1} + \frac{\alpha\phi}{\kappa}\Gamma\left(\frac{1}{\kappa}\right)\sum_{j=0}^{p+4}(-1)^j(j+1)^{-\frac{1}{\kappa}}\binom{p+5}{j+1}, \tag{39}$$

substituted Equations 35 to 39 into Equations 14 to 18 respectively, so

$$\lambda_1^p = \gamma + \frac{2\delta\phi^2}{\kappa}\Gamma\left(\frac{2}{\kappa}\right)\sum_{j=0}^{p}(-1)^j(j+1)^{-\frac{2}{\kappa}}\binom{p+1}{j+1} + \frac{\alpha\phi}{\kappa}\Gamma\left(\frac{1}{\kappa}\right)\sum_{j=0}^{p}(-1)^j(j+1)^{-\frac{1}{\kappa}}\binom{p+1}{j+1}, \tag{40}$$

$$\lambda_2^p = \frac{p+2}{2!}\left[\frac{2\delta\phi^2}{\kappa}\Gamma\left(\frac{2}{\kappa}\right)\sum_{j=0}^{p+1}(-1)^j(j+1)^{-\frac{2}{\kappa}}\binom{p+1}{j} + \frac{\alpha\phi}{\kappa}\Gamma\left(\frac{1}{\kappa}\right)\sum_{j=0}^{p+1}(-1)^j(j+1)^{-\frac{1}{\kappa}}\binom{p+1}{j}\right], \tag{41}$$

$$\lambda_3^p = \frac{p+3}{3!}\left[\frac{2\delta\phi^2}{\kappa}\Gamma\left(\frac{2}{\kappa}\right)\sum_{j=0}^{p+2}(-1)^j(j+1)^{-\frac{2}{\kappa}}(j+2)\binom{p+2}{j} + \frac{\alpha\phi}{\kappa}\Gamma\left(\frac{1}{\kappa}\right)\sum_{j=0}^{p+2}(-1)^j(j+1)^{-\frac{1}{\kappa}}(j+2)\binom{p+2}{j}\right], \tag{42}$$

$$\lambda_4^p = \frac{p+4}{4!}\left[\frac{2\delta\phi^2}{\kappa}\Gamma\left(\frac{2}{\kappa}\right)\sum_{j=0}^{p+3}(-1)^j(j+1)^{-\frac{2}{\kappa}}(j+2)(j+3)\binom{p+3}{j} + \frac{\alpha\phi}{\kappa}\Gamma\left(\frac{1}{\kappa}\right)\sum_{j=0}^{p+3}(-1)^j(j+1)^{-\frac{1}{\kappa}}(j+2)(j+3)\binom{p+3}{j}\right], \tag{43}$$

$$\lambda_5^p = \frac{p+5}{5!}\begin{bmatrix}\frac{2\delta\phi^2}{\kappa}\Gamma\left(\frac{2}{\kappa}\right)\sum_{j=0}^{p+4}(-1)^j(j+1)^{-\frac{2}{\kappa}}(j+2)(j+3)(j+4)\binom{p+4}{j} \\ +\frac{\alpha\phi}{\kappa}\Gamma\left(\frac{1}{\kappa}\right)\sum_{j=0}^{p+4}(-1)^j(j+1)^{-\frac{1}{\kappa}}(j+2)(j+3)(j+4)\binom{p+4}{j}\end{bmatrix} \tag{44}$$

The formulas for the LH-moment ratios, including the LH-moment coefficient of variation (τ_2^p), LH-moment skewness (τ_3^p), and LH-moment kurtosis (τ_4^p), can be obtained by substituting the corresponding LH-moments $\lambda_1^p, \lambda_2^p, \lambda_3^p$ and λ_4^p into Equations 9 to 11, respectively. The LH-moment-based parameter estimation procedure for the W4DN is presented in the following section.

4-2- LH-Moments Parameter Estimation of W4DN

The main advantage of LH-moments is that parameter estimates are more accurate than by method of moments, especially for small samples. For the sample sizes of data (n), the sample may be ranked in ascending order as $y_{(1)} \le y_{(2)} \le \ldots \le y_{(n)}$. The $\hat{\lambda}_r^p$, is unbiased estimator of λ_r^p, which is proved by Wang [22] as; for $p = 0, 1, 2, \ldots$ and $r = 1, 2, 3, 4, 5$,

$$\hat{\lambda}_1^p = \frac{1}{C_{p+1}^n}\sum_{i=1}^n C_p^{i-1} y_{(i)},$$

$$\hat{\lambda}_2^p = \frac{1}{2C_{p+2}^n}\sum_{i=1}^n \left[C_{p+1}^{i-1} - C_p^{i-1}C_1^{n-1}\right] y_{(i)},$$

$$\hat{\lambda}_3^p = \frac{1}{3C_{p+3}^n}\sum_{i=1}^n \left[C_{p+2}^{i-1} - 2C_{p+1}^{i-1}C_1^{n-1} + C_p^{i-1}C_2^{n-1}\right] y_{(i)},$$

$$\hat{\lambda}_4^p = \frac{1}{4C_{p+4}^n}\sum_{i=1}^n \left[C_{p+3}^{i-1} - 3C_{p+2}^{i-1}C_1^{n-1} + 3C_{p+1}^{i-1}C_2^{n-1} - C_p^{i-1}C_3^{n-1}\right] y_{(i)},$$

$$\hat{\lambda}_5^p = \frac{1}{5C_{p+5}^n}\sum_{i=1}^n \left[C_{p+4}^{i-1} - 4C_{p+3}^{i-1}C_1^{n-1} + 6C_{p+2}^{i-1}C_2^{n-1} - 4C_{p+1}^{i-1}C_3^{n-1} + C_p^{i-1}C_4^{n-1}\right] y_{(i)},$$

where $C_j^i = \frac{i!}{j!(i-j)!}$.

The sample LH-moments ratios can be derived from Equation 9 to 11 to yield

sample LH-moments of CV: $\hat{\tau}_2^p = \frac{\hat{\lambda}_2^p}{\hat{\lambda}_1^p}$,

sample LH-moments of skewness: $\hat{\tau}_3^p = \frac{\hat{\lambda}_3^p}{\hat{\lambda}_2^p}$,

sample LH-moments of kurtosis: $\hat{\tau}_4^p = \frac{\hat{\lambda}_4^p}{\hat{\lambda}_2^p}$.

The difficulties in practical applications is parameter estimates of LH-moments. Therefore, Varadhan [31] proposed an augmented Lagrangian adaptive barrier minimization algorithm to optimize for a non-linear solution with non-linear constraints using the statistical software R.

The objective function to arrive at zero is defined as

$$f(\alpha,\delta,\phi,\kappa) = (\lambda_2^p - \hat{\lambda}_2^p)^2 + (\lambda_3^p - \hat{\lambda}_3^p)^2 + (\lambda_4^p - \hat{\lambda}_4^p)^2 + (\lambda_5^p - \hat{\lambda}_5^p)^2 \geq 0 \tag{45}$$

The parameter estimates ($\hat{\alpha}, \hat{\delta}, \hat{\phi}$ and $\hat{\kappa}$) can be found from the solution of Equation 45. Moreover, the estimates of $\hat{\gamma}$ is obtained by plugging $\hat{\alpha}, \hat{\delta}, \hat{\phi}$ and $\hat{\kappa}$ into λ_1^p, as follows,

From Equation 26, in case of $\delta > 0$,

$$\hat{\gamma} = \hat{\lambda}_1^p - \frac{(\hat{\delta}\hat{\phi}^2+\hat{\alpha}\hat{\phi})}{\hat{\kappa}}\Gamma\left(\frac{1}{\hat{\kappa}}\right)\sum_{j=0}^{p}(-1)^j(j+1)^{-\frac{1}{\hat{\kappa}}}\binom{p+1}{j+1}.$$

From Equation 40, in case of $\delta < 0$,

$$\hat{\gamma} = \hat{\lambda}_1^p - \frac{2\hat{\delta}\hat{\phi}^2}{\hat{\kappa}}\Gamma\left(\frac{2}{\hat{\kappa}}\right)\sum_{j=0}^{p}(-1)^j(j+1)^{-\frac{2}{\hat{\kappa}}}\binom{p+1}{j+1} - \frac{\hat{\alpha}\hat{\phi}}{\hat{\kappa}}\Gamma\left(\frac{1}{\hat{\kappa}}\right)\sum_{j=0}^{p}(-1)^j(j+1)^{-\frac{1}{\hat{\kappa}}}\binom{p+1}{j+1}.$$

5- Conclusion

Recently, data has been used to predict extreme events, such as extreme rainfall, extreme temperature, floods, droughts and height of waves for applications in fields such as meteorology, hydrology, agriculture, and engineering. Mostly, the distributions of extreme value are generalized extreme value distribution, generalized Pareto distribution, Kappa distribution, Wakeby distribution, and Weibull distribution.

Many researchers have studied and developed these distributions, including Weibull distribution. The Weibull distribution is commonly used for modeling phenomena with monotonic failure rates and lifetime data. The most important behavior of the Weibull distribution involves nonlinear behavior data in order to forecast the extreme characteristic of the process, which should accurately reflect the tail of the distribution. Several parameter estimation methods have been widely used, such as the Maximum Likelihood Estimation (MLE) and the L-moments method, with studies comparing their performance for various distributions [19, 20]. In addition, LH-moments and PL-moments have also been proposed for parameter estimation. For instance, Murshed et al. [24] investigated the LH-moment estimation for the four-parameter Kappa distribution, while Busababodhin et al. [25] examined LH-moment estimation for the Wakeby distribution. Moreover, Guayjarernpanishk et al. [32] studied the PL-moments for estimating parameters of the four-parameter Kappa distribution. Therefore, the formulas of estimate parameters of four-parameter Weibull for weakly non-linear random variables based on LH-moments method are derived. The new proposed formulas have been derived in two cases as follows:

In case of $\delta > 0$, the r-th LH-moments with order $p = 0, 1, 2, \ldots$ for $r = 1, 2, 3, 4, 5$ are as follows.

$$\lambda_1^p = \gamma + \frac{(\delta\phi^2+\alpha\phi)}{\kappa}\Gamma\left(\frac{1}{\kappa}\right)\sum_{j=0}^{p}(-1)^j(j+1)^{-\frac{1}{\kappa}}\binom{p+1}{j+1},$$

$$\lambda_2^p = \frac{p+2}{2!}\frac{(\delta\phi^2+\alpha\phi)}{\kappa}\Gamma\left(\frac{1}{\kappa}\right)\sum_{j=0}^{p+1}(-1)^j(j+1)^{-\frac{1}{\kappa}}\binom{p+1}{j},$$

$$\lambda_3^p = \frac{p+3}{3!}\frac{(\delta\phi^2+\alpha\phi)}{\kappa}\Gamma\left(\frac{1}{\kappa}\right)\sum_{j=0}^{p+2}(-1)^j(j+1)^{-\frac{1}{\kappa}}(j+2)\binom{p+2}{j},$$

$$\lambda_4^p = \frac{p+4}{4!}\frac{(\delta\phi^2+\alpha\phi)}{\kappa}\Gamma\left(\frac{1}{\kappa}\right)\sum_{j=0}^{p+3}(-1)^j(j+1)^{-\frac{1}{\kappa}}(j+2)(j+3)\binom{p+3}{j},$$

$$\lambda_5^p = \frac{p+5}{5!}\frac{(\delta\phi^2+\alpha\phi)}{\kappa}\Gamma\left(\frac{1}{\kappa}\right)\sum_{j=0}^{p+4}(-1)^j(j+1)^{-\frac{1}{\kappa}}(j+2)(j+3)(j+4)\binom{p+4}{j}.$$

Therefore, the formulas of estimate parameters are optimized by augmented Lagrangian adaptive barrier minimization algorithm to optimize for a non-linear solution with non-linear constraints. So, the parameter estimates ($\hat{\alpha}, \hat{\delta}, \hat{\varphi}$ and $\hat{\kappa}$) are derived. The estimates of $\hat{\gamma}$ can be calculated by

$$\hat{\gamma} = \hat{\lambda}_1^p - \frac{(\hat{\delta}\hat{\phi}^2+\hat{\alpha}\hat{\phi})}{\hat{\kappa}}\Gamma\left(\frac{1}{\hat{\kappa}}\right)\sum_{j=0}^{p}(-1)^j(j+1)^{-\frac{1}{\hat{\kappa}}}\binom{p+1}{j+1},$$

In a similar way, in case of $\delta < 0$, the r-th LH-moments with order $p = 0, 1, 2, \ldots$ for $r = 1, 2, 3, 4, 5$ are as follows.

$$\lambda_1^p = \gamma + \frac{2\delta\phi^2}{\kappa}\Gamma\left(\frac{2}{\kappa}\right)\sum_{j=0}^{p}(-1)^j(j+1)^{-\frac{2}{\kappa}}\binom{p+1}{j+1} + \frac{\alpha\phi}{\kappa}\Gamma\left(\frac{1}{\kappa}\right)\sum_{j=0}^{p}(-1)^j(j+1)^{-\frac{1}{\kappa}}\binom{p+1}{j+1},$$

$$\lambda_2^p = \frac{p+2}{2!}\left[\frac{2\delta\phi^2}{\kappa}\Gamma\left(\frac{2}{\kappa}\right)\sum_{j=0}^{p+1}(-1)^j(j+1)^{-\frac{2}{\kappa}}\binom{p+1}{j} + \frac{\alpha\phi}{\kappa}\Gamma\left(\frac{1}{\kappa}\right)\sum_{j=0}^{p+1}(-1)^j(j+1)^{-\frac{1}{\kappa}}\binom{p+1}{j}\right],$$

$$\lambda_3^p = \frac{p+3}{3!}\left[\frac{2\delta\phi^2}{\kappa}\Gamma\left(\frac{2}{\kappa}\right)\sum_{j=0}^{p+2}(-1)^j(j+1)^{-\frac{2}{\kappa}}(j+2)\binom{p+2}{j} + \frac{\alpha\phi}{\kappa}\Gamma\left(\frac{1}{\kappa}\right)\sum_{j=0}^{p+2}(-1)^j(j+1)^{-\frac{1}{\kappa}}(j+2)\binom{p+2}{j}\right],$$

$$\lambda_4^p = \frac{p+4}{4!}\left[\frac{2\delta\phi^2}{\kappa}\Gamma\left(\frac{2}{\kappa}\right)\sum_{j=0}^{p+3}(-1)^j(j+1)^{-\frac{2}{\kappa}}(j+2)(j+3)\binom{p+3}{j} + \frac{\alpha\phi}{\kappa}\Gamma\left(\frac{1}{\kappa}\right)\sum_{j=0}^{p+3}(-1)^j(j+1)^{-\frac{1}{\kappa}}(j+2)(j+3)\binom{p+3}{j}\right],$$

$$\lambda_5^p = \frac{p+5}{5!}\begin{bmatrix}\frac{2\delta\phi^2}{\kappa}\Gamma\left(\frac{2}{\kappa}\right)\sum_{j=0}^{p+4}(-1)^j(j+1)^{-\frac{2}{\kappa}}(j+2)(j+3)(j+4)\binom{p+4}{j} \\ +\frac{\alpha\phi}{\kappa}\Gamma\left(\frac{1}{\kappa}\right)\sum_{j=0}^{p+4}(-1)^j(j+1)^{-\frac{1}{\kappa}}(j+2)(j+3)(j+4)\binom{p+4}{j}\end{bmatrix}.$$

The estimates of $\hat{\gamma}$ can be calculated by

$$\hat{\gamma} = \hat{\lambda}_1^p - \frac{2\hat{\delta}\hat{\phi}^2}{\hat{\kappa}}\Gamma\left(\frac{2}{\hat{\kappa}}\right)\sum_{j=0}^{p}(-1)^j(j+1)^{-\frac{2}{\hat{\kappa}}}\binom{p+1}{j+1} - \frac{\hat{\alpha}\hat{\phi}}{\hat{\kappa}}\Gamma\left(\frac{1}{\hat{\kappa}}\right)\sum_{j=0}^{p}(-1)^j(j+1)^{-\frac{1}{\hat{\kappa}}}\binom{p+1}{j+1}.$$

Moreover, the newly derived formulas of estimate parameters can be applied to accurately forecast return levels of the extreme events. The principal advantage of this research is its ability to estimate parameters from datasets with small sample sizes. The results obtained can be applied across a wide range of disciplines. For instance, in engineering, particularly irrigation engineering, the proposed formulas can be employed to forecast maximum water levels in reservoirs. This facilitates the effective planning and design of dam structures to ensure they can safely accommodate water inflow and prevent overtopping. In the field of flood management, the formula aids in the prediction of extreme rainfall events, providing essential data for planning and implementing effective flood mitigation strategies. In meteorology, it enables the forecasting of extreme weather events, thereby enhancing disaster preparedness. Moreover, the findings can support the development of agricultural insurance schemes by offering data-driven insights into weather-related risks. For future research, this parameter estimation formula will be applied to real events data, for example, extreme flood data, temperature data, wave data, wind data, and so on.

6- Declarations

6-1- Author Contributions

Conceptualization, P.G. and M.C.; methodology, P.G. and M.C.; validation, P.G.; formal analysis, P.G. and M.C.; investigation, P.G.; writing—original draft preparation, P.G. and M.C.; writing—review and editing, P.G. and M.C.; funding acquisition, P.G. and M.C. All authors have read and agreed to the published version of the manuscript.

6-2- Data Availability Statement

Data sharing is not applicable to this article.

6-3- Funding and Acknowledgments

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6-4- Institutional Review Board Statement

Not applicable.

6-5- Informed Consent Statement

Not applicable.

6-6- Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this manuscript. In addition, the ethical issues, including plagiarism, informed consent, misconduct, data fabrication and/or falsification, double publication and/or submission, and redundancies have been completely observed by the authors.

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