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Bayesian Approaches for Poisson Distribution Parameter Estimation

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Abstract

The Bayesian approach, a non-classical estimation technique, is very widely used in statistical inference for real world situations. The parameter is considered to be a random variable, and knowledge of the prior distribution is used to update the parameter estimation. Herein, two Bayesian approaches for Poisson parameter estimation by deriving the posterior distribution under the squared error loss or quadratic loss functions are proposed. Their performances were compared with frequentist (maximum likelihood estimator) and Empirical Bayes approaches through Monte Carlo simulations. The mean square error was used as the test criterion for comparing the methods for point estimation; the smallest value indicates the best performing method with the estimated parameter value closest to the true parameter value. Coverage Probabilities (CPs) and average lengths (ALs) were obtained to evaluate the performances of the methods for constructing confidence intervals. The results reveal that the Bayesian approaches were excellent for point estimation when the true parameter value was small ($\theta = 0.5$, 1 and 2). In the credible interval comparison, these methods obtained CP values close to the nominal 0.95 confidence level and the smallest ALs for large sample sizes ($n = 50$ and 100), when the true parameter value was small ($\theta = 0.5$, 1 and 2).

Keywords:

Bayesian Method;
Empirical Bayes Approach;
Poisson Distribution;
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1- Introduction

The Poisson distribution plays an important role in the statistical analysis of count data .This type of data arises from situations in which there are several opportunities for the event of interest to occur, such as the number of customers calling a help center in a day, visitors to a net idol YouTube channel, patients infected with Covid-19 per day, and so on .Therefore, the Poisson distribution can be used to determine the probability of several events in a particular time period.

Various researchers have developed inference procedures for a Poisson distribution. Araveeporn [1] proposed inferential statistics for testing hypotheses by using the mean of a Poisson parameter estimator obtained via the maximum likelihood estimator (MLE), Markov Chain-Monte Carlo, and Bayesian approaches. Hassan et al. [2] investigated Bayesian and MLE estimators for a zero-truncated Poisson distribution. Bayesian estimators for a Poisson distribution using a natural conjugate prior [3] and under linex loss function [4-6] and different symmetric and asymmetric loss functions (squared error, linex, precautionary and general entropy) [7] have also been presented. As well as the Poisson distribution, the Bayesian technique for parameter estimation has been extended to the geometric distribution [8], binomial distribution [9], Pareto distribution [10], exponential distribution family [11, 12], double exponential distribution under symmetric and asymmetric loss functions [13], gamma distribution under generalized weighted loss

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function [14], under precautionary loss function [15] and under entropy loss function [16], Weibull distribution [16], inverse Weibull distribution [18], Rayleigh distribution [19], delta-lognormal distribution [20-22], the Lomax distribution base on type-II censored data [23] and using a Uniform and Jeffery prior under different loss functions [24], and inverse Lomax distribution under progressive type-II censoring scheme [25]. The results mostly indicate that the Bayesian-based estimates using the different loss functions were close to the true values, thereby suggesting that this could be a useful approach for estimating parameters of interest in various situations.

The Empirical Bayesian (EB) approach introduced by Robbins [26] is an approximation of the more exact Bayesian approach for a certain sample size. EB is independent with a fixed but unknown parameter of the prior distribution. Estimating this parameter by analyzing the current data is the first step in the EB method. This is not a typical or pure Bayesian method since the parameter of the prior is specified and estimating the hyperprior must be carried out via a classical method such as MLE or the method of moments.

Many researchers have studied Bayesian and EB methods in various situations. Supharakonsaun and Jampachasri [27] introduced an EB estimator for a Poisson distribution by using an exponential distribution that is a special case of a gamma distribution under the squared-error loss function as the posterior marginal distribution; MLE was used to estimate its hyperparameter, which was extended for estimating the hyperparameter via a resampling technique. Mohammed [28] reported expected Bayesian and expected EB approaches for estimating the unknown parameter of a Poisson model under the squared-error loss function. The findings from these two studies indicate that Bayesian and EB estimators are more effective than classical estimators. The efficacy of EB has been compared with hierarchical Bayesian estimation for the unknown parameter of a Poisson distribution under the entropy loss function [29]; the expression of EB is simpler and it performed better than hierarchical Bayesian estimation.

Popular classical methods for estimating the hyperparameter of the prior distribution are MLE and the method of moments. Estimating the Poisson parameter via the Bayesian posterior distribution under Stein's method or the squared-error loss function and an EB estimator with a conjugate gamma prior showed that the MLE for hyperparameter estimation is better than the method of moments estimator for the EB approach [30]. Similarly, the case of a normal distribution with a conjugated normal-inverse-gamma prior was computed by Zhang et al. [31].

Approaches for estimating the parameter of a random variable can be grouped into 3 categories: classical, Bayesian, and empirical Bayesian (EB), which are all useful under different sets of circumstances. The Bayesian methods provide estimates of the unknown parameter by using a fixed informative prior distribution, are better than classical approaches for most situations involving loss functions. Therefore, estimating a Poisson parameter using Bayesian methods under different loss functions is of interest in the present study.

Our motivation for this study is to estimate the Poisson parameter by using MLE, Bayesian, and EB methods for both point and interval estimation. The main goal of the paper is focused the Bayesian estimation of Poisson parameter under the squared error and quadratic loss functions. Derivations of these approaches are covered in Section 2. The results of a simulation study are reported in Section 3. Finally, the conclusions based on our findings are presented in the last section.

2- Estimation Methods for Mean of Poisson Distribution

2-1- The Maximum Likelihood Estimator (MLE)

MLE is a simple method for constructing an estimator for the unknown parameter of a probability distribution by maximizing a likelihood function. It can be applied to a wide variety of statistical problems and provides a reasonable and excellent estimator for the parameter when the sample size is large.

Suppose $X_1, X_2, \dots, X_n$ are random variables with a probability mass function from independent and identically distributed random variables from a Poisson distribution with parameter θ .The probability function of $X_i; i = 1, 2, \dots, n$ denoted by $f(x_i|\theta)$ can be derived as follows:

$$f(x_i \mid \theta) = \frac{e^{-\theta}\theta^{x_i}}{x_i!};\ x_i = 0,1,2,\dots;\ \theta > 0, \quad (1)$$

where θ is constant mean rate of event occur.

The joint probability mass function or product of *n* terms is called likelihood function, which is defined as;

$$L(\underline{X} \mid \theta) = \prod_{i=1}^{n} f(x_i \mid \theta) = \frac{e^{-n\theta}\theta^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!}. \quad (2)$$

In order to make the joint probability mass function to be a monotonic function, the logarithm is taken to the likelihood function. That is;

$$\ln L\left(\underline{X} \mid \theta\right) = -n\theta + \sum_{i=1}^{n} x_i \ln\theta - \ln\prod_{i=1}^{n} x_i\,! . \tag{3}$$

From the logarithm of the likelihood function, the maximum of $\ln L(\underline{X}|\theta)$ occurs at the same value of θ as does the maximum of $L(\underline{X}|\theta)$. If $\ln L(\underline{X}|\theta)$ is differentiable in parameter θ, the necessary conditions for the occurrence of a maximum is solved by applying;

$$\frac{\partial \ln L\left(\underline{X} \mid \lambda\right)}{\partial\theta} = -n + \frac{\sum_{i=1}^{n} x_i}{\theta} = 0 . \tag{4}$$

By the solving of necessary condition as mentioned above, we can obtain the MLE of θ as

$$\hat{\theta}_{MLE} = \frac{\sum_{i=1}^{n} x_i}{n} = \overline{x} . \tag{5}$$

Because we do not know if this is a maximum or minimum value, the second derivative of the estimator can be used to prove that the estimator is the maximum when the second derivative is less than 0 as follows:

$$\frac{\partial^2 \ln L\left(\underline{X} \mid \lambda\right)}{\partial\theta^2} = -\frac{\sum_{i=1}^{n} x_i}{\theta^2} < 0 . \tag{6}$$

Therefore, the MLE of θ is $\hat{\theta}_{MLE} = \bar{x}$.

2-2- Bayesian Method

To estimate the Bayesian estimator, the prior probability distribution under gamma prior is specified by using squared error and quadratic loss functions.

Let $X_1, X_2, \dots, X_n$ be random variables from a Poisson distribution. The probability mass function of the random variable is given by $f(x_i|\theta)$ with the likelihood function $L(\theta) = \prod_{i=1}^{n} f(x_i|\theta)$. Consider the informative conjugate prior for θ is a Gamma distribution with parameters a, b where a is the shape parameter and b is the scale parameter. It is given by,

$$h\left(\theta \mid a,b\right) = \frac{b^a}{\Gamma\left(a\right)}\theta^{a-1}e^{-b\theta},\ \ a,b,\theta > 0 . \tag{7}$$

The posterior distribution for Bayesian procedure can be derived by considering the combining of the likelihood function (2) and the prior distribution (7) as follows:

$$f\left(\underline{x} \mid \theta\right)\pi\left(\theta\right) = \frac{b^a e^{-(n+b)\theta} \cdot \theta^{\sum_{i=1}^{n} x_i + a - 1}}{\prod_{i=1}^{n} x_i\,!\,\Gamma\left(a\right)} \tag{8}$$

The marginal probability density function of θ can be derived by integration of the combing of the likelihood function and the prior distribution. It can be derived as follows:

$$\int_0^\infty f\left(\underline{x} \mid \theta\right)\pi\left(\theta\right)d\theta = \int_0^\infty \frac{b^a e^{-(n+b)\theta} \cdot \theta^{\sum_{i=1}^{n} x_i + a - 1}}{\prod_{i=1}^{n} x_i\,!\,\Gamma\left(a\right)} d\theta$$

$$= \frac{b^a\,\Gamma\left(\sum_{i=1}^{n} x_i + a\right)}{\prod_{i=1}^{n} x_i\,!\,\Gamma\left(a\right)\left(n+b\right)^{\sum_{i=1}^{n} x_i + a}} \int_0^\infty \frac{\left(n+b\right)^{\sum_{i=1}^{n} x_i + a} e^{-(n+b)\theta} \cdot \theta^{\sum_{i=1}^{n} x_i + a - 1}}{\Gamma\left(\sum_{i=1}^{n} x_i + a\right)} d\theta = \frac{b^a\,\Gamma\left(\sum_{i=1}^{n} x_i + a\right)}{\prod_{i=1}^{n} x_i\,!\,\Gamma\left(a\right)\left(n+b\right)^{\sum_{i=1}^{n} x_i + a}}$$

Now, the posterior distribution of θ is derived by;

$$h(\theta\,|\,\underline{x}) = \frac{\dfrac{b^a e^{-(n+b)\theta}\cdot\theta^{\sum_{i=1}^{n}x_i+a-1}}{\prod_{i=1}^{n}x_i!\,\Gamma(a)}}{\dfrac{b^a\,\Gamma\left(\sum_{i=1}^{n}x_i+a\right)}{\prod_{i=1}^{n}x_i!\,\Gamma(a)(n+b)^{\sum_{i=1}^{n}x_i+a}}} = \frac{(n+b)^{\sum_{i=1}^{n}x_i+a}\,\theta^{\sum_{i=1}^{n}x_i+a-1}\,e^{-(n+b)\theta}}{\Gamma\left(\sum_{i=1}^{n}x_i+a\right)}. \tag{9}$$

This implies that the posterior distribution can be written as;

$$h(\theta\,|\,\underline{x}) = \frac{(n+b)^{\sum_{i=1}^{n}x_i+a}}{\Gamma\left(\sum_{i=1}^{n}x_i+a\right)}\theta^{\sum_{i=1}^{n}x_i+a-1}e^{-(n+b)\theta}, \tag{10}$$

which is a gamma distribution with parameters $\sum_{i=1}^{n}x_i + a$ and $n + b$. Hence,

$$\theta\,|\,\underline{X} \sim Gamma\left(\sum_{i=1}^{n}x_i+a,\, n+b\right). \tag{11}$$

2-2-1- The Bayesian Estimator of Parameter θ for Squared Error (SE) Loss Function

The Bayesian estimator for θ for the squared error loss function that is defined as;

$$L(\hat{\theta};\theta) = \left(\hat{\theta}-\theta\right)^2. \tag{12}$$

The squared error loss function of the Bayesian estimator is the mean of the posterior distribution function, which can be derived as;

$$\hat{\theta}_{BS} = E(\theta\,|\,\underline{x})$$

$$= \frac{(n+b)^{\sum_{i=1}^{n}x_i+a}}{\Gamma\left(\sum_{i=1}^{n}x_i+a\right)}\int_0^\infty \theta\cdot\theta^{\sum_{i=1}^{n}x_i+a-1}e^{-(n+b)\theta}d\theta = \frac{(n+b)^{\sum_{i=1}^{n}x_i+a}\,\Gamma\left(\sum_{i=1}^{n}x_i+a+1\right)}{\Gamma\left(\sum_{i=1}^{n}x_i+a\right)(n+b)^{\sum_{i=1}^{n}x_i+a+1}}\int_0^\infty \frac{(n+b)^{\sum_{i=1}^{n}x_i+a+1}\,\theta^{\sum_{i=1}^{n}x_i+a}\,e^{-(n+b)\theta}}{\Gamma\left(\sum_{i=1}^{n}x_i+a+1\right)}d\theta$$

$$= \frac{(n+b)^{\sum_{i=1}^{n}x_i+a}\,\Gamma\left(\sum_{i=1}^{n}x_i+a+1\right)}{\Gamma\left(\sum_{i=1}^{n}x_i+a\right)(n+b)^{\sum_{i=1}^{n}x_i+a+1}} = \frac{\sum_{i=1}^{n}x_i+a}{n+b}.$$

The Bayesian estimator of θ under the squared error loss function is shown by,

$$\hat{\theta}_{BS} = \frac{\sum_{i=1}^{n}x_i+a}{n+b}. \tag{13}$$

2-2-2- The Bayesian Estimator of Parameter θ for Quadratic Loss (QL) Function

The quadratic loss function which is a non-negative symmetric and continuous loss function of parameter θ and estimate of $\hat{\theta}$ for the Bayesian estimator is defined as [12];

$$L(\hat{\theta};\theta) = \left(\frac{\hat{\theta}-\theta}{\theta}\right)^2. \tag{14}$$

The Bayesian estimator under quadratic loss function of θ is obtained by deriving the following equation:

$$\frac{\partial}{\partial\hat{\theta}}\int L\left(\hat{\theta};\theta\right)h\left(\theta\,|\,\underline{x}\right)d\theta=0\,. \tag{15}$$

The Bayesian estimator for parameter θ of the Poisson distribution under a quadratic loss function can be solved by;

$$\frac{\partial}{\partial\hat{\theta}}\int\left(\frac{\hat{\theta}-\theta}{\theta}\right)^2\frac{(n+b)^{\sum_{i=1}^{n}x_i+a}}{\Gamma\left(\sum_{i=1}^{n}x_i+a\right)}\theta^{\sum_{i=1}^{n}x_i+a-1}e^{-(n+b)\theta}\,d\theta=0$$

$$\int\frac{2\left(\hat{\theta}-\theta\right)}{\theta^2}\frac{(n+b)^{\sum_{i=1}^{n}x_i+a}}{\Gamma\left(\sum_{i=1}^{n}x_i+a\right)}\theta^{\sum_{i=1}^{n}x_i+a-1}e^{-(n+b)\theta}\,d\theta=0$$

$$\frac{(n+b)^{\sum_{i=1}^{n}x_i+a}}{\Gamma\left(\sum_{i=1}^{n}x_i+a\right)}\hat{\theta}\int\theta^{\left(\sum_{i=1}^{n}x_i+a-2\right)-1}e^{-(n+b)\theta}\,d\theta=\frac{(n+b)^{\sum_{i=1}^{n}x_i+a}}{\Gamma\left(\sum_{i=1}^{n}x_i+a\right)}\int\theta^{\left(\sum_{i=1}^{n}x_i+a-1\right)-1}e^{-(n+b)\theta}\,d\theta$$

$$\frac{(n+b)^{\sum_{i=1}^{n}x_i+a}}{\Gamma\left(\sum_{i=1}^{n}x_i+a\right)}\hat{\theta}\frac{\Gamma\left(\sum_{i=1}^{n}x_i+a-2\right)}{(n+b)^{\sum_{i=1}^{n}x_i+a-2}}=\frac{(n+b)^{\sum_{i=1}^{n}x_i+a}}{\Gamma\left(\sum_{i=1}^{n}x_i+a\right)}\frac{\Gamma\left(\sum_{i=1}^{n}x_i+a-1\right)}{(n+b)^{\sum_{i=1}^{n}x_i+a-1}}$$

$$\hat{\theta}=\frac{\Gamma\left(\sum_{i=1}^{n}x_i+a-1\right)}{(n+b)^{\sum_{i=1}^{n}x_i+a-1}}\frac{(n+b)^{\sum_{i=1}^{n}x_i+a-2}}{\Gamma\left(\sum_{i=1}^{n}x_i+a-2\right)}=\frac{\sum_{i=1}^{n}x_i+a-2}{n+b}$$

The Bayesian estimator of θ under a quadratic loss function is expressed as;

$$\hat{\theta}_{QL}=\frac{\sum_{i=1}^{n}x_i+a-2}{n+b}\,. \tag{16}$$

2-3- The Empirical Bayes (EB) Estimation

The idea behind the EB method is different from the Bayesian approach .To estimate the true parameter value by using the Bayesian approach, the hyperparameter is assumed to be known or if unknown, prior information on it is available .The hyperparameter is independent of the observations .With the EB method, the first step of estimating the hyperparameter is by using the classical approach of assessing the observations, after which the hyperparameter is substituted by the estimation in the posterior distribution.

Let $X_1, X_2, \ldots, X_n$ be random variables from a Poisson distribution. The probability mass function of the random variable is given by (1):

$$f\left(x_i\,|\,\theta\right)=\frac{e^{-\theta}\theta^{x_i}}{x_i\,!},\,x_i=0,1,2,\ldots\,.$$

Supharakonsakun and Jampachasri [27] proposed the EB estimator by studying the case of θ as ab exponential distribution denoted by $\theta \sim Exp(\lambda)$. Hence, the prior distribution function of θ can be written as;

$$\pi\left(\theta\right)=\lambda e^{-\lambda\theta}\quad;\,\theta,\,\lambda>0\,. \tag{17}$$

Subsequently, the posterior marginal distribution of X can be solved by;

$$m(x_i\,|\,\theta)=\int_0^\infty f(x\,|\,\lambda)\pi(\theta)d\theta\,. \tag{18}$$

This is derived as follows:

$$m(x_i\,|\,\lambda)=\int_0^\infty \frac{e^{-\theta}\theta^{x_i}}{x_i\,!}\lambda e^{-\lambda\theta}d\theta=\frac{\lambda}{x_i\,!}\frac{\Gamma(x_i+1)}{(\lambda+1)^{x_i+1}}\int_0^\infty\frac{(\theta+1)^{x_i+1}}{\Gamma(x_i+1)}\theta^{(x_i+1)-1}e^{-(\lambda+1)\theta}d\theta=\frac{\lambda}{x_i\,!}\frac{\Gamma(x_i+1)}{(\lambda+1)^{x_i+1}}$$

$$m(x_i\,|\,\lambda)=\frac{\lambda}{(\lambda+1)^{x_i+1}}\quad ;x_i=0,1,2,\dots.$$

We now obtain the posterior marginal distribution of X as a geometric distribution denoted by $X|\lambda\sim GEO\left(\frac{\lambda}{\lambda+1}\right)$, which can be rewritten in the form;

$$m(x_i\,|\,\lambda)=\left(\frac{\lambda}{\lambda+1}\right)\left(1-\frac{\lambda}{\lambda+1}\right)^{x_i-1}\quad ;x_i=1,2,3,\dots. \tag{19}$$

Hyperparameter λ is estimated in the next step of the EB procedure. The MLE is used to estimate λ by considering the likelihood function of the posterior marginal distribution as follows:

$$L(\underline{X}\,|\,\lambda)=\prod_{i=1}^{n}\frac{\lambda}{(\lambda+1)^{x_i+1}}=\frac{\lambda^n}{(\lambda+1)^{\sum_{i=1}^{n}x_i+n}}\,. \tag{20}$$

The logarithm of the likelihood function is;

$$\ln L(\underline{X}\,|\,\lambda)=n\ln\lambda-\left(\sum_{i=1}^{n}x_i+n\right)\ln(\lambda+1)\,. \tag{21}$$

From the logarithm of the likelihood function, MLE for λ is solved by;

$$\frac{\partial\ln L(\underline{X}\,|\,\lambda)}{\partial\lambda}=\frac{n}{\lambda}-\frac{\sum_{i=1}^{n}x_i+n}{\lambda+1}=0\,. \tag{22}$$

We obtain the MLE of the hyperparameter λ as;

$$\hat{\lambda}_{MLE}=\frac{n}{\sum_{i=1}^{n}x_i}=\frac{1}{\overline{x}}\,. \tag{23}$$

The posterior distribution of the EB procedure can be derived as the product of the likelihood function and the prior distribution as follows:

$$f(\underline{x}\,|\,\theta)\pi(\theta)=\frac{\lambda e^{-(n+\lambda)\theta}\cdot\theta^{\sum_{i=1}^{n}x_i}}{\prod_{i=1}^{n}x_i\,!}\,. \tag{24}$$

The integration of the product produces;

$$\int_0^\infty f(\underline{x}\,|\,\theta)\pi(\theta)d\theta=\int_0^\infty\frac{\lambda e^{-(n+\lambda)\theta}\cdot\theta^{\sum_{i=1}^{n}x_i}}{\prod_{i=1}^{n}x_i\,!}d\theta=\frac{\lambda\Gamma\left(\sum_{i=1}^{n}x_i+1\right)}{\prod_{i=1}^{n}x_i\,!(n+\lambda)^{\sum_{i=1}^{n}x_i+1}}\int_0^\infty\frac{(n+\lambda)^{\sum_{i=1}^{n}x_i+1}}{\Gamma\left(\sum_{i=1}^{n}x_i+1\right)}e^{-(n+\lambda)\theta}\cdot\theta^{\left(\sum_{i=1}^{n}x_i+1\right)-1}d\theta\,.$$

Thus,

$$\int_0^\infty f\left(\underline{x}\mid\theta\right)\pi(\theta)\,d\theta=\frac{\lambda\Gamma\left(\sum_{i=1}^{n}x_i+1\right)}{\prod_{i=1}^{n}x_i!\,(n+\lambda)^{\sum_{i=1}^{n}x_i+1}}. \tag{25}$$

The posterior distribution of θ given $\underline{x}$ becomes;

$$h\left(\theta\mid\underline{x}\right)=\frac{\dfrac{\lambda e^{-(n+\lambda)\theta}\cdot\theta^{\sum_{i=1}^{n}x_i}}{\prod_{i=1}^{n}x_i!}}{\dfrac{\lambda\Gamma\left(\sum_{i=1}^{n}x_i+1\right)}{\prod_{i=1}^{n}x_i!\,(n+\lambda)^{\sum_{i=1}^{n}x_i+1}}}=\frac{(n+\lambda)^{\sum_{i=1}^{n}x_i+1}\cdot e^{-(n+\lambda)\theta}\cdot\theta^{\sum_{i=1}^{n}x_i}}{\Gamma\left(\sum_{i=1}^{n}x_i+1\right)}. \tag{26}$$

We now have the posterior distribution of $\underline{x}$ that is the Gamma distribution denoted by;

$$\theta\mid\underline{X}\sim Gamma\left(\sum_{i=1}^{n}x_i+1,n+\lambda\right). \tag{27}$$

After we have obtained the posterior distribution, we then substitute the hyperparameter λ with estimator $\hat{\lambda}$ in the posterior distribution of $\underline{x}$. Hence, we obtain;

$$\theta\mid\underline{X}\sim Gamma\left(\sum_{i=1}^{n}x_i+1,n+\hat{\lambda}\right). \tag{28}$$

Therefore, the EB estimator of θ under the squared error loss function is given by;

$$\hat{\theta}_{EB}=\frac{\sum_{i=1}^{n}x_i+1}{n+\hat{\lambda}}. \tag{29}$$

Moreover, point estimation of hyperparameter is extended via a bootstrapping method for 1,000 iterations. Thus, there are two methods to estimate the true parameter θ using the EB procedure (EB and EB with bootstrappings).

3- Simulation Results

A simulation study with 5,000 replications was conducted to estimate the performance of point estimates and confidence intervals constructed with the 5 methods. For point estimation, the lowest mean-squared error (MSE) was the criterion used to identify the best-performing method. Besides, for comparing the confidence intervals constructed with them, coverage probabilities (CPs) and average lengths (Als) were calculated under the same conditions of varying hyperparameters a and b for the Bayesian approaches in the point estimation. A CP close to or greater than the nominal level of 0.95 and the shortest AL were used to identify the best-performing method in each case. The Monte Carlo simulation study is presented in Figure 1.

3-1- Point Estimation

The point estimation performances of the MLE, Bayesian under the squared-error loss function (B_{SL}), Bayesian under quadratic loss function (B_{QL}), EB, and EB with bootstrapping (EB_{boot}) methods for estimating the mean of a Poisson distribution were compared via simulation. Random samples were implemented to generate 5,000 sets from a Poisson distribution and 1,000 iterations for bootstrapping the hyperparameter estimation for EB_{boot} with various sample sizes (n = 5, 10, 15, 20, 30, 50, 100) and true parameter θ = 0.5, 1, 2, 5, 10, 20 given arbitrary prior parameter (a,b) = (2,2), (2,4), (1,3.5) and (4,4) for Bayesian methods under two different loss functions. The mean-squared error (MSE) is the criterion used to evaluate the performances of the parameter estimation methods, which is computed as

$$MSE=\frac{\sum_{t=1}^{m}\left(\theta-\hat{\theta}_t\right)}{m}. \tag{30}$$

The lowest MSE value means that the estimated value of θ is closest to its true value.

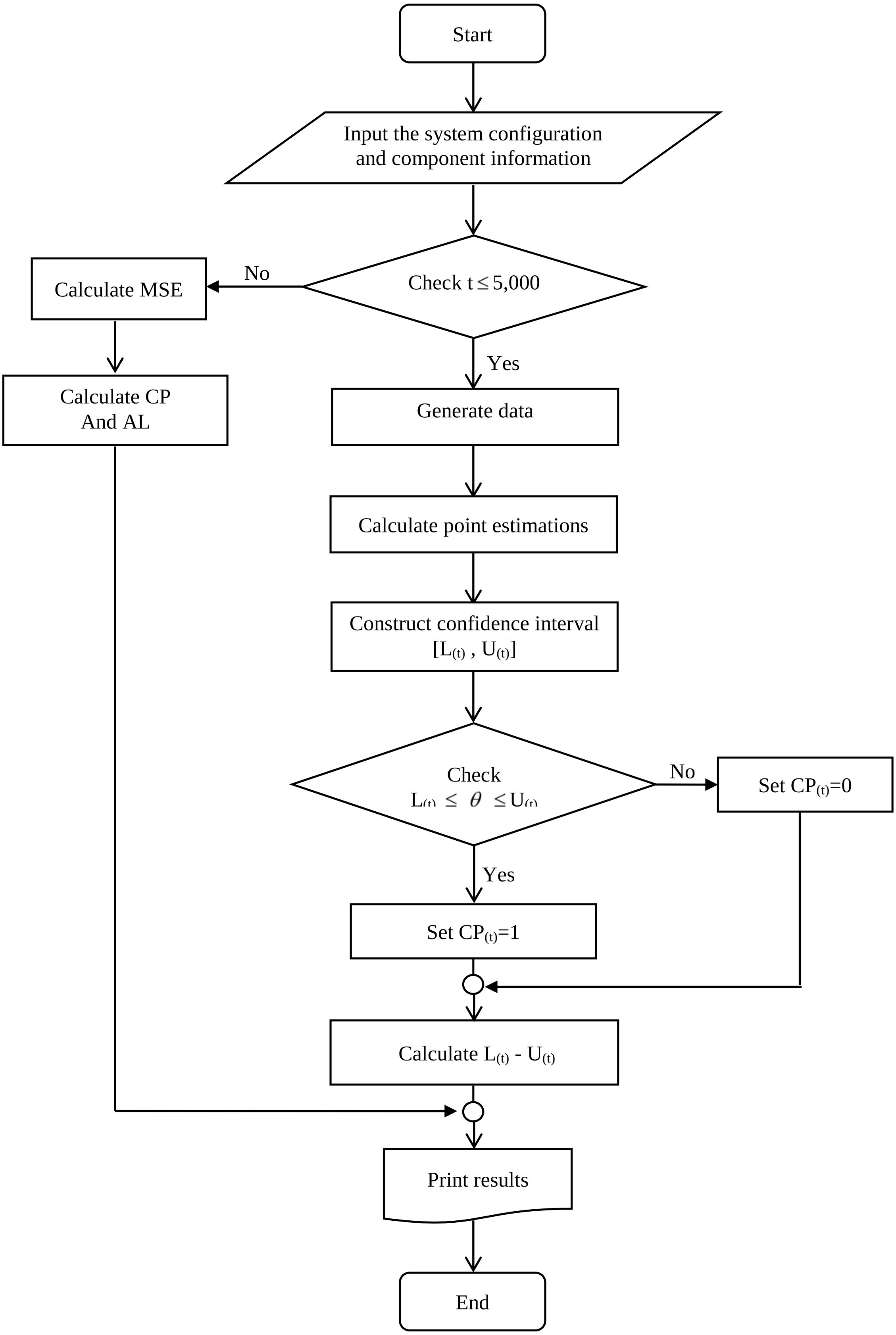

Figure 1. Monte Carlo simulation flowchart.

Table 1 reports the MSE values for point estimation using MLE, B_{SL}, B_{QL}, EB, and EB_{boot}, with hyperparameters $a = 2$ and $b = 2$ for the Bayesian methods. For all magnitudes of sample sizes n, B_{SL} outperformed the others with the lowest MSE for true parameter $\theta = 0.5$, 1, or 2 while MLE was the best for $\theta = 5$, 10, or 20. When focusing on the EB approaches, we found that the EB and EB_{boot} performances were very close to each other for all values of hyperparameters tested. Although the EB approaches did not provide the lowest MSEs, they were very close to those of the best estimator in each case.

Table 2 summarizes the MSE values obtained by the five methods, with hyperparameter $a = 2$ and $b = 4$ for the Bayesian methods. For sample sizes $n = 5$, 10, 15, or 20, the lowest MSE values were obtained by B_{SL} for $\theta = 0.5$ or 1; by B_{QL} for $\theta = 2$; by MLE for $\theta = 5$, 10, or 20. For sample sizes $n = 30$, 50, or 100, the lowest MSE values were obtained by B_{SL} for $\theta = 0.5$; by B_{QL} for $\theta = 1$ or 2; and by MLE for $\theta = 5$, 10, or 20. Meanwhile, EB and EB_{boot} yielded low MSEs close to the minimum MSE in each case.

The results in Table 3 comprise MSE values obtained by the five methods, with hyperparameters $a = 1$ and $b = 3.5$ for the Bayesian methods. This time, the lowest MSE values were obtained by B_{SL} for $\theta = 0.5$; by B_{QL} for $\theta = 1$ or 2; and by MLE for $\theta = 5$, 10, or 20 for all variations in sample size n. Meanwhile, EB and EB_{boot} provided consistently low MSEs that were only slightly higher than the others.

The MSE results in Table 4 for the five methods (with hyperparameters $a = 4$ and $b = 4$ for the Bayesian methods) indicate that for sample sizes $n = 5$ or 10, the lowest MSE values were obtained by B_{SL} for $\theta = 0.5$ or 1; by B_{QL} for $\theta = 2$; and by MLE for $\theta = 5$, 10, or 20. For sample sizes $n = 15$, 20, 30, 50, or 100, the lowest MSE values were obtained by B_{SL} for $\theta = 1$, by B_{QL} for $\theta = 2$, and by MLE for $\theta = 0.5$, 5, 10, or 20. Although EB and EB_{boot} performed worse than the others, their MSEs were close to the lowest MSE in each case.

Figure 2 exhibits the point estimation performances of the MLE, B_{SL}, B_{QL}, EB, and EB_{boot} methods in terms of MSE for various sample sizes. The results show that the MSE values of all methods increased with increasing θ but decreased with increasing sample size. All five point estimation methods provided similar MSE values for $\theta = 0.5$, 1, 2, or 5 and all magnitudes of sample size. For $\theta = 10$ and 20, the MSE values obtained with B_{SL} exceeded the values obtained with the other methods, and especially for small sample sizes ($n = 5$, 10, 15, 20, or 30), the MSE values from the B_{SL} method increased dramatically. This means that the point estimation performance of B_{SL} was weaker when parameter θ was large. On the other hand, when θ was small (0.5, 1, or 2), B_{SL} outperformed the other methods (Tables 1 to 4).

Table 1. MSE of different estimators of Poisson distribution when *a*=2, *b*=2.

n	θ	Estimators				
		MLE	**B_{SL}**	**B_{QL}**	**EB**	**EB_{Boot}**
5	0.5	0.099104	0.091252	0.303076	0.113081	0.110512
	1	0.201176	0.130994	0.420673	0.235998	0.232976
	2	0.390496	0.326934	0.664979	0.461788	0.467555
	5	1.02084	1.904548	1.389500	1.200347	1.211237
	10	2.001792	7.893585	2.544969	2.381971	2.399838
	20	4.090784	31.606460	5.110772	4.852462	4.836759
10	0.5	0.049380	0.029017	0.045979	0.054428	0.053113
	1	0.096838	0.075536	0.075940	0.106640	0.105970
	2	0.196300	0.182111	0.258166	0.217606	0.214408
	5	0.499204	0.824395	0.598437	0.545319	0.551117
	10	0.989210	3.004691	1.147334	1.076708	1.089071
	20	2.008994	11.615900	2.240245	2.220490	2.206249
15	0.5	0.033924	0.031941	0.053973	0.035728	0.035843
	1	0.068073	0.056555	0.090823	0.072484	0.072344
	2	0.132362	0.125391	0.159198	0.140963	0.142610
	5	0.349068	0.510380	0.385848	0.377020	0.366932
	10	0.685924	1.646210	0.746185	0.736236	0.727646
	20	0.976786	3.849628	1.030730	1.035781	1.029384
20	0.5	0.024987	0.024233	0.037006	0.026291	0.025897
	1	0.048071	0.041569	0.059630	0.050687	0.050784
	2	0.099339	0.094697	0.114555	0.103102	0.104932
	5	0.250262	0.345230	0.277183	0.261497	0.263243
	10	0.486903	1.072216	0.524965	0.514545	0.513949
	20	1.016642	3.917869	1.080546	1.059799	1.057904
30	0.5	0.016293	0.015818	0.021430	0.016794	0.016898
	1	0.033214	0.030335	0.038148	0.034263	0.034054
	2	0.064005	0.062170	0.071148	0.066232	0.066094
	5	0.163170	0.211318	0.174088	0.167863	0.168760
	10	0.338277	0.613403	0.356621	0.350991	0.348555
	20	0.646079	2.022373	0.673131	0.669031	0.665133

50	0.5	0.009847	0.009557	0.011452	0.010040	0.009996
	1	0.019904	0.018929	0.021741	0.020203	0.020323
	2	0.039772	0.038868	0.041909	0.040349	0.041014
	5	0.098478	0.114958	0.102239	0.100793	0.100468
	10	0.201525	0.319802	0.206486	0.206050	0.205508
	20	0.398703	0.909311	0.407968	0.408042	0.404783
100	0.5	0.004860	0.004838	0.005317	0.004899	0.004906
	1	0.010231	0.009943	0.010748	0.010319	0.010281
	2	0.019386	0.019273	0.019927	0.019588	0.019606
	5	0.051752	0.056191	0.052534	0.052242	0.052582
	10	0.100112	0.128760	0.101749	0.101165	0.100730
	20	0.198776	0.328405	0.201395	0.200816	0.201257

Table 2. MSE of different estimators of Poisson distribution when *a*=2, *b*=4.

n	θ	Estimators				
		MLE	**B_{SL}**	**B_{QL}**	**EB**	**EB_{Boot}**
5	0.5	0.193056	0.124405	0.130901	0.226791	0.234690
	1	0.194896	0.125003	0.127759	0.225084	0.228800
	2	0.389568	0.605250	0.330863	0.467007	0.454938
	5	0.987616	4.432340	1.967464	1.192460	1.170282
	10	2.016640	18.658890	7.903486	2.387870	2.448923
	20	4.028128	76.554400	32.105830	4.814068	4.903727
10	0.5	0.050622	0.029126	0.046228	0.054692	0.054099
	1	0.096402	0.075086	0.075646	0.106676	0.105394
	2	0.202824	0.299329	0.185010	0.223185	0.221703
	5	0.501806	1.935992	0.822112	0.552956	0.550252
	10	1.012412	8.024528	3.095003	1.120841	1.103653
	20	1.987440	32.197900	11.526990	2.180303	2.188647
15	0.5	0.033326	0.022549	0.031063	0.035105	0.035181
	1	0.066933	0.056037	0.056983	0.069727	0.071655
	2	0.136877	0.193247	0.128325	0.144046	0.144282
	5	0.333276	1.089243	0.485680	0.354739	0.361134
	10	0.669480	4.439124	1.679942	0.720296	0.714857
	20	1.353934	17.746350	6.134013	1.447406	1.436745
20	0.5	0.025197	0.018496	0.023837	0.026225	0.026087
	1	0.049681	0.042573	0.043099	0.052011	0.051669
	2	0.098797	0.134372	0.093235	0.103673	0.103373
	5	0.251608	0.749857	0.352579	0.264320	0.264829
	10	0.473187	2.865952	1.085616	0.492836	0.494520
	20	1.011615	11.252580	3.821647	1.059451	1.054805
30	0.5	0.016795	0.013526	0.016360	0.017247	0.017373
	1	0.033546	0.030866	0.030633	0.034612	0.034714
	2	0.032744	0.030045	0.029769	0.033704	0.033724
	5	0.162496	0.416173	0.214092	0.167617	0.168261
	10	0.330746	1.504826	0.610592	0.343417	0.341874
	20	0.65508	5.765841	1.994743	0.677318	0.676764
50	0.5	0.010090	0.008862	0.009862	0.010323	0.010201
	1	0.020328	0.019206	0.019139	0.020668	0.020729
	2	0.039910	0.047207	0.039120	0.040642	0.040870
	5	0.100505	0.196560	0.117192	0.102418	0.102749
	10	0.201708	0.672322	0.311498	0.204678	0.207057
	20	0.404519	2.468836	0.931205	0.410182	0.412696

100	0.5	0.004917	0.004575	0.004850	0.004981	0.004976
	1	0.010109	0.009868	0.009833	0.010220	0.010209
	2	0.019633	0.021411	0.019333	0.019793	0.019885
	5	0.049455	0.074940	0.053606	0.049950	0.049847
	10	0.100845	0.231952	0.131336	0.101846	0.101993
	20	0.196926	0.737712	0.326533	0.198606	0.198981

Table 3. **MSE of different estimators of Poisson distribution when a=1, b=3.5.**

n	θ	Estimators				
		MLE	B_{SL}	B_{QL}	EB	EB_{Boot}
5	0.5	0.098816	0.050447	0.078236	0.113964	0.112924
	1	0.208144	0.177792	0.159188	0.239794	0.246682
	2	0.382824	0.649632	0.365832	0.450044	0.458498
	5	0.995288	4.206471	1.730495	1.167655	1.168030
	10	2.014328	16.810190	6.126311	2.383784	2.448520
	20	4.103576	67.551330	22.761530	4.884260	4.838685
10	0.5	0.050112	0.034032	0.043090	0.054152	0.053884
	1	0.101258	0.093709	0.086043	0.110660	0.111894
	2	0.205040	0.321132	0.199921	0.221530	0.224819
	5	0.497482	1.823945	0.748054	0.547680	0.542045
	10	0.998318	6.933701	2.320960	1.090330	1.087383
	20	2.080266	27.346320	8.020482	2.277475	2.261644
15	0.5	0.033710	0.025487	0.029654	0.035561	0.035911
	1	0.064460	0.063921	0.058381	0.069243	0.068932
	2	0.131875	0.195334	0.130171	0.138891	0.141515
	5	0.332070	1.024969	0.446691	0.359863	0.355129
	10	0.669570	3.886330	1.334571	0.717756	0.710855
	20	1.324652	14.932590	4.288395	1.396016	1.418584
20	0.5	0.024829	0.020238	0.022701	0.025765	0.025932
	1	0.048869	0.048007	0.045057	0.050567	0.050542
	2	0.098541	0.140529	0.098200	0.104324	0.103266
	5	0.253787	0.673977	0.319856	0.264830	0.267035
	10	0.501309	2.483548	0.878890	0.526486	0.521736
	20	0.977366	9.331528	2.686589	1.031685	1.039903
30	0.5	0.016378	0.014188	0.015409	0.016722	0.016807
	1	0.032973	0.032750	0.031430	0.034075	0.034026
	2	0.066745	0.087901	0.066593	0.069067	0.069040
	5	0.166107	0.382418	0.200666	0.171428	0.171133
	10	0.325179	1.291212	0.495207	0.339034	0.334482
	20	0.664770	4.744609	1.447919	0.689368	0.688527
50	0.5	0.010057	0.009258	0.009560	0.010235	0.010219
	1	0.019962	0.020229	0.019227	0.020242	0.020416
	2	0.039810	0.047227	0.039513	0.040570	0.040687
	5	0.099396	0.179104	0.109629	0.101381	0.101704
	10	0.197786	0.586977	0.268928	0.202314	0.202123
	20	0.413135	2.029550	0.715810	0.419022	0.420516
100	0.5	0.005020	0.004768	0.004930	0.005074	0.005073
	1	0.009843	0.009873	0.009673	0.009872	0.009956
	2	0.019963	0.022189	0.019815	0.020108	0.020250
	5	0.051148	0.073086	0.054014	0.051684	0.051919
	10	0.101070	0.204408	0.118711	0.101971	0.102117
	20	0.205896	0.644675	0.285679	0.207335	0.207835

Table 4. MSE of different estimators of Poisson distribution when *a*=4, *b*=4.

n	θ	Estimators				
		MLE	B_{SL}	B_{QL}	EB	EB_{Boot}
5	0.5	0.100128	0.098743	0.263371	0.114641	0.115051
	1	0.195960	0.083822	0.222572	0.230449	0.231624
	2	0.405232	0.356165	0.263037	0.478373	0.473320
	5	1.010904	3.516952	1.350870	1.202746	1.207529
	10	1.999648	16.702050	6.392148	2.419265	2.430623
	20	3.969584	72.647800	28.971470	4.771550	4.831326
10	0.5	0.051434	0.050878	0.102122	0.056908	0.055128
	1	0.101432	0.059270	0.107722	0.110767	0.110743
	2	0.198226	0.193303	0.153180	0.218184	0.217774
	5	0.491562	1.581361	0.636411	0.534177	0.540410
	10	0.974770	7.124140	2.490179	1.079565	1.077697
	20	2.064076	31.090050	10.826180	2.290927	2.236347
15	0.5	0.033396	0.033792	0.058953	0.035357	0.035628
	1	0.066196	0.044897	0.070442	0.069794	0.070221
	2	0.128922	0.134221	0.108821	0.136559	0.138378
	5	0.333124	0.920332	0.394654	0.357071	0.353144
	10	0.687593	3.964010	1.416239	0.729413	0.734068
	20	1.350105	16.838740	5.567503	1.422659	1.428900
20	0.5	0.024077	0.024649	0.039265	0.025165	0.025444
	1	0.051149	0.037625	0.053279	0.054460	0.053690
	2	0.101576	0.104136	0.088298	0.104857	0.107049
	5	0.252716	0.628713	0.292682	0.265554	0.267322
	10	0.506098	2.624467	0.968324	0.534303	0.534119
	20	0.995604	10.862430	3.581061	1.050006	1.035090
30	0.5	0.016563	0.016616	0.023824	0.017149	0.017237
	1	0.034366	0.027512	0.034932	0.035673	0.035712
	2	0.065286	0.067196	0.059680	0.067438	0.067591
	5	0.167061	0.359689	0.188329	0.172232	0.171931
	10	0.346848	1.398092	0.567188	0.356259	0.358324
	20	0.681776	5.544798	1.885139	0.711597	0.702547
50	0.5	0.009799	0.009962	0.012544	0.009994	0.009961
	1	0.019790	0.017343	0.020198	0.020098	0.020137
	2	0.040080	0.040739	0.037857	0.040819	0.041019
	5	0.100763	0.175407	0.108698	0.103098	0.102907
	10	0.198738	0.619418	0.282222	0.202806	0.204290
	20	0.394630	2.298448	0.834769	0.403044	0.400599
100	0.5	0.005159	0.005187	0.005922	0.005212	0.005225
	1	0.010127	0.009450	0.010150	0.010248	0.010252
	2	0.019272	0.019336	0.018720	0.019515	0.019455
	5	0.050189	0.071657	0.052608	0.050808	0.050569
	10	0.099895	0.214220	0.121308	0.100803	0.100747
	20	0.196378	0.714174	0.314145	0.198759	0.198714

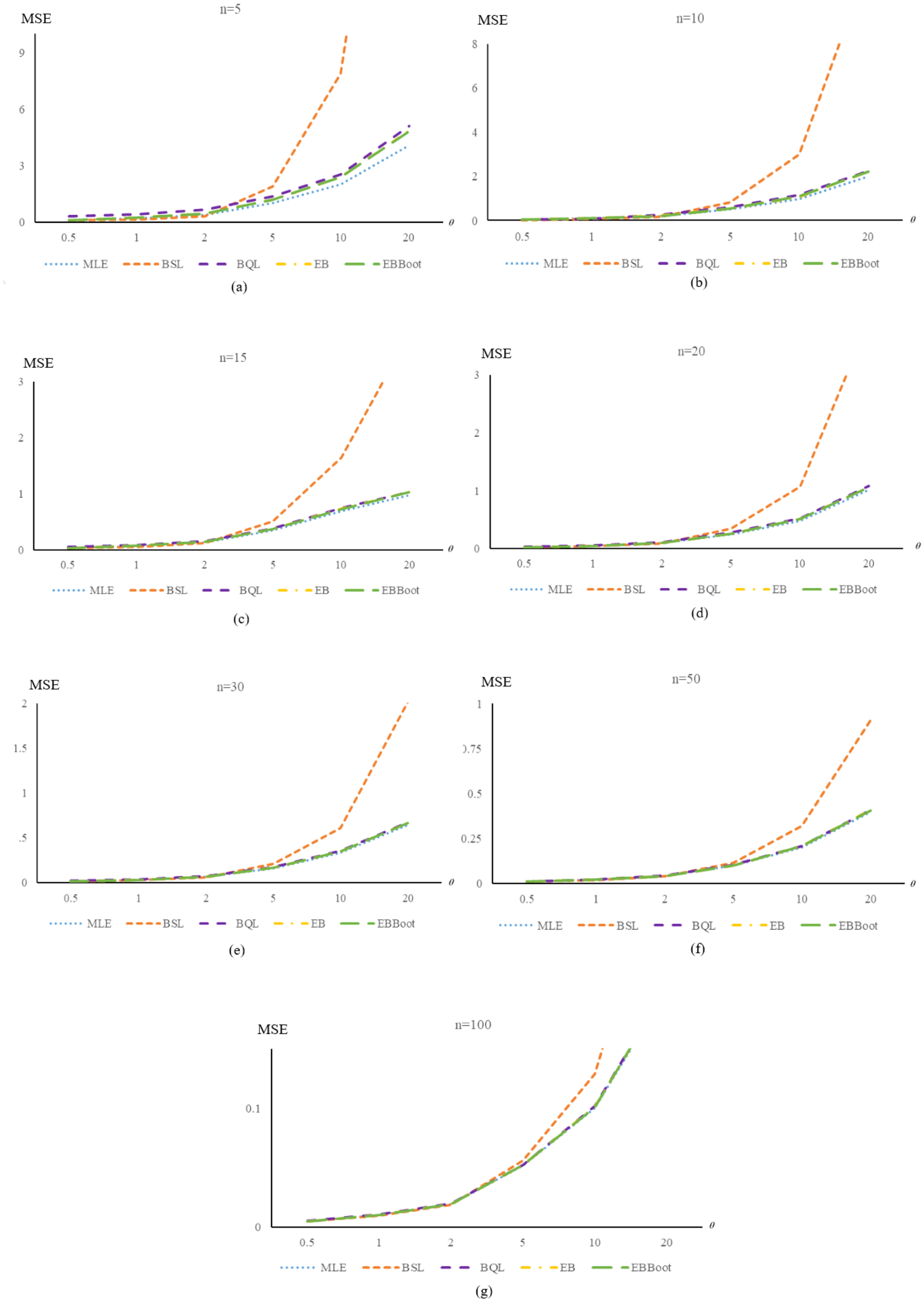

Figure 2. Comparative performance of point estimation.

3-2- Confident Interval

The parameter settings for comparing the confidence interval performances were the same as for the point estimation simulation. Coverage probabilities (CPs) and average lengths (AL) were used to evaluate the performances of the methods. The most effective confidence interval method obtained a CP close to or greater than the nominal level of 0.95 and the shortest AL.

Table 5. Coverage probability (CP) and average length (AL) of 95% of confidence interval for the mean of Poisson distribution when *a*=2, *b*=2.

n	θ	CP					AL				
		MLE	B_{SL}	B_{QL}	EB	EB_{Boot}	MLE	B_{SL}	B_{QL}	EB	EB_{Boot}
5	0.5	0.9106	0.6996	0.5494	0.5256	0.5386	1.607555	0.632610	0.890093	0.518620	0.517925
	1	0.9580	0.6942	0.5926	0.5900	0.6024	2.412134	0.799737	1.120636	0.844432	0.844703
	2	0.9706	0.5944	0.6180	0.6168	0.6116	3.470401	1.068726	1.482813	1.285845	1.288048
	5	0.9864	0.3504	0.6340	0.6316	0.6316	5.520415	1.617572	2.270936	2.122267	2.128895
	10	0.9926	0.1404	0.6366	0.6394	0.6340	7.834095	2.251841	3.126911	3.067146	3.056830
	20	0.9932	0.0210	0.6456	0.6430	0.6380	11.08278	3.152468	4.439264	4.358594	4.311047
10	0.5	0.9584	0.8258	0.7516	0.7192	0.7186	0.975306	0.603393	0.725451	0.542678	0.544237
	1	0.9718	0.8138	0.7594	0.7568	0.7642	1.420385	0.810666	0.962627	0.835642	0.838655
	2	0.9548	0.7528	0.7694	0.7804	0.7734	2.006209	1.094355	1.309834	1.225022	1.219264
	5	0.9720	0.5920	0.7836	0.7760	0.7840	3.190490	1.712794	2.031309	1.965699	1.979353
	10	0.9762	0.3774	0.7834	0.7866	0.7894	4.519046	2.383543	2.846706	2.814343	2.832078
	20	0.9760	0.1518	0.8004	0.7820	0.7890	6.391771	3.350430	4.038289	3.975973	3.989187
15	0.5	0.9404	0.8688	0.8206	0.8040	0.8002	0.769913	0.557150	0.631012	0.518728	0.516031
	1	0.9578	0.8640	0.8376	0.8288	0.8326	1.097136	0.750590	0.851667	0.771445	0.772412
	2	0.9542	0.8222	0.8352	0.8324	0.8400	1.559227	1.032362	1.167479	1.118811	1.116834
	5	0.9668	0.7148	0.8386	0.8378	0.8442	2.470452	1.613843	1.834007	1.782541	1.800720
	10	0.9718	0.5342	0.8434	0.8554	0.8436	3.499240	2.261647	2.570372	2.544284	2.550873
	20	0.9614	0.2990	0.8430	0.8408	0.8394	4.952375	3.217632	3.638149	3.623931	3.608942
20	0.5	0.9280	0.8904	0.8522	0.8408	0.8452	0.651502	0.510694	0.558601	0.482467	0.485399
	1	0.9516	0.8798	0.8518	0.8510	0.8542	0.929861	0.697142	0.765758	0.710648	0.714416
	2	0.9570	0.8562	0.8678	0.8592	0.8604	1.317382	0.964032	1.064626	1.023904	1.023173
	5	0.9628	0.7694	0.8674	0.8704	0.8700	2.090040	1.512135	1.658402	1.637905	1.640942
	10	0.9624	0.6430	0.8716	0.8706	0.8718	2.960116	2.131572	2.336404	2.320729	2.320040
	20	0.9622	0.4158	0.8766	0.8746	0.8750	4.185683	3.007765	3.314869	3.301599	3.293631
30	0.5	0.9224	0.9130	0.8858	0.8794	0.8848	0.501376	0.440308	0.472163	0.424875	0.426768
	1	0.9278	0.8990	0.8860	0.8850	0.8834	0.714164	0.611624	0.650244	0.620488	0.618455
	2	0.9522	0.8886	0.8910	0.8920	0.8928	1.009908	0.849029	0.910523	0.884960	0.884097
	5	0.9520	0.8432	0.8986	0.8966	0.8926	1.600797	1.333246	1.427179	1.413425	1.412316
	10	0.9516	0.7438	0.8972	0.8986	0.8974	2.262729	1.876996	2.004556	2.002416	2.002127
	20	0.9510	0.5780	0.8938	0.8984	0.8918	3.200934	2.657712	2.837162	2.832057	2.828042
50	0.5	0.9500	0.9286	0.9054	0.9042	0.9072	0.390226	0.358741	0.371205	0.351463	0.351061
	1	0.9480	0.9188	0.9108	0.9082	0.9058	0.552758	0.497707	0.517736	0.502480	0.502273
	2	0.9434	0.9088	0.9152	0.9090	0.9124	0.782867	0.700523	0.726801	0.717418	0.717098
	5	0.9494	0.8764	0.9148	0.9152	0.9112	1.239173	1.098198	1.144472	1.138241	1.140552
	10	0.9478	0.8218	0.9194	0.9176	0.9150	1.753233	1.552512	1.620128	1.605415	1.613848
	20	0.9516	0.7204	0.9188	0.9218	0.9202	2.478789	2.199163	2.287030	2.279935	2.281450
100	0.5	0.9450	0.9318	0.9216	0.9260	0.9210	0.276829	0.264330	0.269640	0.262297	0.261567
	1	0.9456	0.9372	0.9312	0.9314	0.9318	0.391768	0.371713	0.377911	0.371974	0.373523
	2	0.9464	0.9306	0.9346	0.9310	0.9302	0.554062	0.523320	0.532123	0.528149	0.529758
	5	0.9428	0.9104	0.9258	0.9230	0.9268	0.876227	0.823006	0.840717	0.837201	0.839843
	10	0.9462	0.8852	0.9276	0.9284	0.9288	1.239378	1.165336	1.187373	1.185568	1.188253
	20	0.9514	0.8396	0.9370	0.9328	0.9356	1.753159	1.644476	1.682819	1.676053	1.679322

Table 6. Coverage probability (CP) and average length (AL) of 95% of confidence interval for the mean of Poisson distribution when *a*=2, *b*=4.

n	θ	CP					AL				
		MLE	**B_{SL}**	**B_{QL}**	**EB**	**EB_{Boot}**	**MLE**	**B_{SL}**	**B_{QL}**	**EB**	**EB_{Boot}**
5	0.5	0.9178	0.7138	0.6974	0.5454	0.5446	1.619054	0.489907	0.634784	0.522727	0.520517
	1	0.9564	0.5538	0.6746	0.5746	0.5820	2.410230	0.625186	0.805963	0.844115	0.853402
	2	0.9740	0.3116	0.5828	0.6052	0.6006	3.476643	0.829444	1.060156	1.282092	1.298908
	5	0.9906	0.0486	0.3566	0.6258	0.6212	5.529801	1.263263	1.614621	2.129835	2.111677
	10	0.9916	0.0010	0.1338	0.6306	0.6392	7.826954	1.743831	2.241875	2.986772	3.071715
	20	0.9934	0.0000	0.0222	0.6526	0.6394	11.10013	2.47144	3.140614	4.387615	4.38398
10	0.5	0.9634	0.8368	0.8278	0.7276	0.7406	0.982202	0.518259	0.605560	0.551392	0.551751
	1	0.9658	0.7282	0.8142	0.7554	0.7510	1.411626	0.693180	0.801714	0.835429	0.837994
	2	0.9540	0.5428	0.7572	0.7686	0.7690	2.011582	.9402596	1.097768	1.223463	1.227203
	5	0.9708	0.2010	0.5902	0.7762	0.7838	3.188316	1.451263	1.687773	1.970505	1.979217
	10	0.9796	0.0328	0.392	0.7926	0.7888	4.522996	2.050157	2.389078	2.821249	2.816042
	20	0.9774	0.0008	0.1644	0.7826	0.7896	6.39667	2.864764	3.350547	4.023185	4.011785
15	0.5	0.9354	0.8650	0.8612	0.7990	0.7980	0.769246	0.498004	0.553855	0.520145	0.519903
	1	0.9552	0.8088	0.8558	0.8230	0.8154	1.100694	0.673443	0.750170	0.771495	0.772001
	2	0.9580	0.6596	0.8142	0.8372	0.8340	1.558429	0.927173	1.033403	1.119145	1.115776
	5	0.9650	0.3498	0.7032	0.8348	0.8394	2.471100	1.440579	1.611227	1.784248	1.789175
	10	0.9704	0.1042	0.5414	0.8442	0.8496	3.498840	2.028200	2.272177	2.553718	2.537857
	20	0.9666	0.0070	0.3024	0.8366	0.8450	4.950921	2.862376	3.201113	3.597744	3.600471
20	0.5	0.9330	0.8942	0.8854	0.8454	0.8440	0.653778	0.468232	0.511280	0.486629	0.484713
	1	0.9572	0.8464	0.8830	0.8540	0.8614	0.928856	0.635397	0.693782	0.706837	0.712288
	2	0.9588	0.7394	0.8518	0.8574	0.8662	1.321475	0.886235	0.968396	1.025802	1.020940
	5	0.9598	0.4756	0.7702	0.8668	0.8702	2.091889	1.381536	1.508342	1.641463	1.637707
	10	0.9600	0.2064	0.6406	0.8674	0.8626	2.960464	1.956247	2.136977	2.319926	2.319953
	20	0.9634	0.0324	0.4286	0.8722	0.8682	4.186600	2.755330	3.010266	3.290685	3.300505
30	0.5	0.9162	0.9120	0.9024	0.8806	0.8760	0.501499	0.414360	0.440025	0.427111	0.425545
	1	0.9318	0.8838	0.9016	0.8880	0.8876	0.715524	0.574723	0.611604	0.620615	0.621832
	2	0.9496	0.8134	0.8914	0.8946	0.8948	1.010807	0.799951	0.849725	0.884262	0.883668
	5	0.9484	0.5966	0.8264	0.8844	0.8904	1.597668	1.255698	1.329586	1.405938	1.402207
	10	0.9538	0.3632	0.7470	0.9010	0.8958	2.264022	1.769438	1.891281	1.987458	1.994811
	20	0.9478	0.0976	0.5488	0.8986	0.8990	3.198483	2.506965	2.653921	2.825603	2.817794
50	0.5	0.9502	0.9240	0.9248	0.9062	0.9030	0.390971	0.345088	0.359276	0.351181	0.351694
	1	0.9440	0.8980	0.9102	0.9076	0.9066	0.552646	0.480353	0.497871	0.502346	0.503862
	2	0.9426	0.8566	0.9054	0.9116	0.9030	0.783156	0.672858	0.699409	0.718351	0.714519
	5	0.9462	0.7518	0.8844	0.9114	0.9102	1.240324	1.062735	1.103290	1.138516	1.138908
	10	0.9488	0.5432	0.8192	0.9136	0.9182	1.751861	1.499732	1.557404	1.611256	1.611675
	20	0.9472	0.2632	0.7142	0.9176	0.9100	2.478177	2.108174	2.195631	2.282742	2.277332
100	0.5	0.9460	0.9350	0.9334	0.9318	0.9304	0.276062	0.258713	0.263430	0.261331	0.261406
	1	0.9426	0.9264	0.9340	0.9308	0.9252	0.391698	0.363462	0.371062	0.373335	0.372599
	2	0.9482	0.9086	0.934	0.9324	0.9330	0.554335	0.513291	0.523896	0.530182	0.528249
	5	0.9504	0.8448	0.9106	0.9296	0.9300	0.876142	0.808593	0.823742	0.837359	0.838808
	10	0.9508	0.7386	0.8890	0.9342	0.9332	1.238915	1.143413	1.166334	1.189681	1.186145
	20	0.9480	0.5594	0.8332	0.9280	0.9278	1.753316	1.613821	1.646551	1.678923	1.678422

Table 7. Coverage probability (CP) and average length (AL) of 95% of confidence interval for the mean of Poisson distribution when a=1, b=3.5.

n	θ	CP					AL				
		MLE	B_{SL}	B_{QL}	EB	EB_{Boot}	MLE	B_{SL}	B_{QL}	EB	EB_{Boot}
5	0.5	0.9180	0.6172	0.6696	0.5410	0.5392	1.630927	0.452135	0.591007	0.535666	0.528802
	1	0.9592	0.4812	0.6342	0.5858	0.5888	2.404796	0.607332	0.789203	0.837424	0.840202
	2	0.9698	0.2962	0.5728	0.6110	0.6150	3.466534	0.845365	1.090610	1.289298	1.290306
	5	0.9872	0.0660	0.4132	0.6356	0.6340	5.524026	1.300733	1.703790	2.120537	2.111179
	10	0.9908	0.0064	0.2508	0.6296	0.6436	7.833084	1.838392	2.394342	3.049160	3.064594
	20	0.9934	0.0000	0.0796	0.6494	0.6384	11.09148	2.598600	3.393944	4.369718	4.344879
10	0.5	0.9594	0.7684	0.8036	0.7304	0.7282	0.979645	0.495225	0.579723	0.544429	0.545553
	1	0.9692	0.6810	0.7916	0.7606	0.7550	1.409392	0.680688	0.805592	0.837319	0.831482
	2	0.9478	0.5188	0.7296	0.7614	0.7636	2.000704	0.949720	1.107310	1.212492	1.215983
	5	0.9750	0.2434	0.6500	0.7908	0.7882	3.190506	1.498681	1.755422	1.984182	1.965789
	10	0.9732	0.0638	0.5130	0.7746	0.7790	4.516912	2.104345	2.482233	2.814242	2.818787
	20	0.9740	0.0040	0.3014	0.7850	0.7864	6.393036	2.985140	3.500916	3.982631	3.981626
15	0.5	0.9376	0.8304	0.8604	0.7994	0.8016	0.767799	0.478725	0.540265	0.519082	0.516549
	1	0.9556	0.7654	0.8382	0.8186	0.8150	1.095613	0.664168	0.746610	0.771155	0.768141
	2	0.9586	0.6616	0.8166	0.8278	0.8286	1.561913	0.932263	1.056257	1.117456	1.115744
	5	0.9690	0.3968	0.7532	0.8462	0.8490	2.472578	1.472772	1.644812	1.791164	1.791246
	10	0.9736	0.1554	0.6552	0.8520	0.8564	3.499628	2.072363	2.329111	2.544133	2.554489
	20	0.9644	0.0206	0.4546	0.8392	0.8348	4.949642	2.925940	3.292354	3.614396	3.598564
20	0.5	0.9292	0.8600	0.8792	0.8428	0.8462	0.651479	0.452906	0.498307	0.482959	0.484167
	1	0.9568	0.8298	0.8736	0.8626	0.8564	0.931360	0.638110	0.697209	0.714872	0.711225
	2	0.9588	0.7342	0.8572	0.8666	0.8636	1.319243	0.896132	0.977165	1.028626	1.029791
	5	0.9632	0.5130	0.7990	0.8666	0.8690	2.091244	1.405291	1.539872	1.642670	1.643886
	10	0.9612	0.2578	0.7218	0.8740	0.8724	2.957603	1.985019	2.182842	2.318158	2.319276
	20	0.9648	0.0660	0.5724	0.8722	0.8708	4.185874	2.809849	3.064425	3.297706	3.294691
30	0.5	0.9126	0.8838	0.8976	0.8672	0.8714	0.5015533	0.4084347	0.4360317	0.428012	0.427603
	1	0.9292	0.8634	0.8918	0.8872	0.8854	0.711836	0.5705448	0.6079093	0.612869	0.620041
	2	0.9544	0.8020	0.8872	0.8984	0.8960	1.010261	0.807160	0.857172	0.885760	0.883386
	5	0.9524	0.6492	0.8574	0.8952	0.8986	1.598787	1.270823	1.345487	1.407942	1.407048
	10	0.9532	0.4294	0.7966	0.8976	0.8954	2.261131	1.799713	1.904499	1.996044	1.992668
	20	0.9500	0.1778	0.6940	0.8932	0.8928	3.200161	2.538920	2.710243	2.827263	2.826159
50	0.5	0.9506	0.9126	0.9136	0.9016	0.9036	0.390541	0.341258	0.354475	0.351300	0.352386
	1	0.9438	0.8934	0.9110	0.9082	0.9084	0.553265	0.479373	0.498875	0.503006	0.502921
	2	0.9502	0.8658	0.9122	0.9112	0.9152	0.783166	0.678508	0.702246	0.715435	0.718075
	5	0.9534	0.7658	0.8864	0.9150	0.9160	1.239135	1.067899	1.112000	1.137311	1.137578
	10	0.9432	0.6110	0.8532	0.9088	0.9076	1.752746	1.511201	1.568788	1.609130	1.611529
	20	0.9498	0.3630	0.7864	0.9128	0.9182	2.477934	2.131182	2.213501	2.275756	2.280907
100	0.5	0.9508	0.9314	0.9374	0.9310	0.9280	0.276173	0.257769	0.262750	0.261717	0.261530
	1	0.9410	0.9162	0.9276	0.9286	0.9272	0.391393	0.363142	0.370356	0.372511	0.372505
	2	0.9444	0.9066	0.9262	0.9330	0.9282	0.554361	0.514372	0.524701	0.528961	0.529943
	5	0.9478	0.8650	0.9226	0.9330	0.9336	0.876456	0.811610	0.830240	0.839697	0.839246
	10	0.9490	0.7756	0.9010	0.9316	0.9318	1.238900	1.146744	1.170690	1.187179	1.186867
	20	0.9498	0.6302	0.8684	0.9324	0.9328	1.753073	1.622126	1.656973	1.679569	1.679983

Table 8. Coverage probability (CP) and average length (AL) of 95% of confidence interval for the mean of Poisson distribution when *a*=4, *b*=4.

n	θ	CP					AL				
		MLE	B_{SL}	B_{QL}	EB	EB_{Boot}	MLE	B_{SL}	B_{QL}	EB	EB_{Boot}
5	0.5	0.9138	0.6438	0.4156	0.5366	0.5386	1.624976	0.612501	0.779173	0.525449	0.527094
	1	0.9616	0.7268	0.6554	0.5962	0.5832	2.406193	0.717967	0.916662	0.843736	0.856087
	2	0.9682	0.4842	0.6736	0.6010	0.6080	3.465726	0.898523	1.149336	1.281356	1.275973
	5	0.9872	0.0780	0.4468	0.6328	0.6218	5.510437	1.304584	1.653120	2.119507	2.111643
	10	0.9906	0.0038	0.1920	0.6356	0.6282	7.834848	1.780249	2.307838	3.055042	3.058725
	20	0.9930	0.0000	0.0294	0.6490	0.6602	11.092860	2.473280	3.196755	4.354572	4.375683
10	0.5	0.9588	0.7976	0.6674	0.7286	0.7286	0.985056	0.600991	0.700818	0.550822	0.545220
	1	0.9682	0.8480	0.8000	0.7736	0.7668	1.410883	0.748984	0.872630	0.834705	0.828385
	2	0.9560	0.6948	0.8254	0.7852	0.7780	2.009617	0.983333	1.141522	1.224653	1.218208
	5	0.9684	0.2798	0.6826	0.7822	0.7790	3.193513	1.485009	1.734832	1.975854	1.973136
	10	0.9764	0.0414	0.4626	0.7892	0.7830	4.519447	2.058062	2.404176	2.814477	2.804773
	20	0.9754	0.0014	0.1836	0.7990	0.7950	6.396466	2.903937	3.376389	4.018442	4.008764
15	0.5	0.9338	0.8474	0.7600	0.8014	0.7988	0.766632	0.548009	0.613780	0.519500	0.515156
	1	0.9584	0.8754	0.8460	0.8258	0.8202	1.099199	0.711726	0.793186	0.770361	0.770540
	2	0.9608	0.7806	0.8694	0.8356	0.8380	1.560337	0.957583	1.073157	1.113441	1.117606
	5	0.9676	0.4314	0.7786	0.8392	0.8362	2.474110	1.459141	1.637839	1.794852	1.781565
	10	0.9676	0.1454	0.6022	0.8412	0.8406	3.501964	2.037875	2.278511	2.536040	2.542311
	20	0.9672	0.0094	0.3290	0.8492	0.8492	4.950872	2.867101	3.212928	3.611731	3.607749
20	0.5	0.9304	0.8696	0.8086	0.8410	0.8386	0.656562	0.506474	0.556824	0.488674	0.487875
	1	0.9528	0.8942	0.8780	0.8574	0.8570	0.931435	0.668763	0.728887	0.713966	0.709739
	2	0.9580	0.8148	0.8878	0.8678	0.8622	1.317849	0.904041	0.988854	1.020844	1.026129
	5	0.9568	0.5234	0.8044	0.8650	0.8648	2.087496	1.392933	1.518305	1.634716	1.636990
	10	0.9618	0.2318	0.6834	0.8722	0.8658	2.958177	1.965138	2.146741	2.322557	2.319068
	20	0.9590	0.0348	0.452	0.8684	0.8722	4.184803	2.753412	3.014108	3.295433	3.298369
30	0.5	0.9148	0.9022	0.8582	0.8742	0.8802	0.502206	0.440640	0.468760	0.426775	0.427025
	1	0.9348	0.9166	0.9002	0.8932	0.8934	0.712650	0.593461	0.628450	0.617534	0.618670
	2	0.9488	0.8602	0.9058	0.8906	0.8872	1.010039	0.812401	0.862668	0.885048	0.884317
	5	0.9520	0.6728	0.8616	0.8982	0.8928	1.599982	1.264963	1.346935	1.411770	1.405024
	10	0.9478	0.3938	0.7740	0.8926	0.8932	2.261469	1.783701	1.888889	1.992391	1.989169
	20	0.9492	0.1052	0.6054	0.8964	0.9018	3.199392	2.497719	2.657876	2.828264	2.832740
50	0.5	0.9494	0.9190	0.8958	0.9054	0.9008	0.389288	0.357244	0.369245	0.349355	0.349183
	1	0.9512	0.9272	0.9194	0.9138	0.9126	0.552617	0.489920	0.507157	0.502239	0.502421
	2	0.9434	0.8952	0.9182	0.9124	0.9046	0.783597	0.682145	0.706176	0.716969	0.715345
	5	0.9422	0.7752	0.8906	0.9070	0.9070	1.238952	1.064611	1.105489	1.139260	1.136898
	10	0.9484	0.5724	0.8468	0.9138	0.9144	1.751751	1.500611	1.556808	1.608410	1.611586
	20	0.9520	0.2964	0.7364	0.9146	0.9206	2.479200	2.109114	2.191986	2.273197	2.280550
100	0.5	0.9514	0.9366	0.9226	0.9298	0.9324	0.277015	0.264949	0.269998	0.262163	0.262541
	1	0.9470	0.9336	0.9342	0.9366	0.9336	0.391244	0.366662	0.375084	0.372798	0.372697
	2	0.9480	0.9246	0.9340	0.9346	0.9332	0.554255	0.516222	0.525529	0.529987	0.531118
	5	0.9466	0.8776	0.9216	0.9352	0.9282	0.876653	0.812158	0.824949	0.840319	0.837829
	10	0.9500	0.7566	0.8926	0.9362	0.9328	1.238742	1.141032	1.166202	1.186886	1.187039
	20	0.9524	0.5736	0.8500	0.9348	0.9334	1.753047	1.616746	1.647908	1.680374	1.68149

Tables 5 to 8 report the CPs and ALs obtained by the MLE, B_{SL}, B_{QL}, EB, and EB_{boot} methods, with hyperparameters (2,2), (2,4), (1,3.5), and (4,4) for the Bayesian methods, respectively. The results show that for sample sizes $n = 5$, 10, 15, 20, or 30, the MLE method provided CP values close to 0.95 for all values of θ, although its ALs were not the shortest. The results for small sample sizes indicate that when θ increased, the performances of B_{SL} and B_{QL} decreased with CP values lower than the nominal level of 0.95 .Although the CPs of EB and EB_{boot} were not close to 0.95, they improved with increasing of θ. In addition, their AL values were similar to each other and narrower than the other methods for $\theta = 0.5$.

For sample sizes $n = 50$ or 100, MLE once again obtained CP values close to 0.95 for all values of θ, as did B_{SL} and B_{QL} for $\theta = 0.5$, 1, or 2. Meanwhile, the CPs of B_{SL} decreased drastically for large values of θ while those of EB and EB_{boot} were similar to each other and higher than B_{SL} and B_{QL}. Although the ALs obtained by the five methods were similar, slightly shorter ones were provided by EB_{boot} for $\theta = 0.5$ and B_{SL} for $\theta = 1$, 2, 5, 10, or 20 in the case of sample size $n = 50$. Meanwhile, the shortest ALs were provided by EB for $\theta = 0.5$ and B_{SL} for $\theta = 1$, 2, 5, 10, or 20 in the case of sample size $n = 100$.

4- Discussion

Frequentist and Bayesian inference are fundamentally different principles in statistics. Estimating the Poisson parameter using the MLE method as a classical frequentist approach was proposed by Araveeporn [1] and Hassan et al. [2]. The existing EB and EB with bootstrapping approaches were derived by Supharakonsakun and Jampachasri [27]. Empirical Bayes methods are different from the classical Bayesian approach in that the hyperparameters are assumed to be unknown and prior observations are used to estimate them via the classical Bayesian approach. Determining the hyperparameters is important for the posterior distribution and increases the accuracy of the parameter estimation. Two different loss functions, squared-error and quadratic [12], for parameter estimation based on the classical Bayesian approach were used in this study.

The results in this study were similar to those of Srivastava [7] in that the Bayesian point estimation method under the squared-error loss function provided estimates nearer to the true value of the Poisson parameter. Besides, the Bayesian estimators under different loss functions performed better than the classical estimator (MLE) in the case of small and different values of the hyperparameters, which is similar to the findings of Hassan and Baizid [12] and Naji and Rasheed who found that Bayesian estimate parameters under precautionary [15], generalized weighted [14] or entropy [16] loss functions were the better than the classical approaches of MLE and the method of moments.

5- Conclusion

The purpose of this study was to derive the Bayesian posterior distribution with the highest equitailed posterior density interval under two different loss functions for point estimation and constructing credible intervals for estimating the Poisson parameter with a gamma prior distribution .The performances of two Bayesian methods under either the squared-error loss function or the quadratic loss function were compared with the existing classical MLE, EB, and EB with bootstrapping approaches through Monte Carlo simulations.

When analyzing the Poisson parameter distribution, the Bayesian methods created under the squared-error and quadratic loss functions produced the most suitable estimates for small true parameter values (θ = 0.5, 1, or 2) by providing the lowest MSE values for point estimation for all cases of sample size. Moreover, they attained CPs close to the nominal 0.95 confidence level with the lowest ALs for a large sample size (50 or 100). Meanwhile, for all cases of sample size, the classical MLE approach obtained the lowest MSE values for point estimation for large true parameter values (θ = 5, 10, or 20) and provided CPs close to or greater than 0.95 and slightly longer ALs than the Bayesian methods. In addition, although the EB estimation method based on exponential prior distribution did not achieve the best results, they were close to those of the best estimator in each case, and so it is a good alternative for point estimation and confidence interval construction.

6- Declarations

6-1- Data Availability Statement

The data presented in this study are available on request from the corresponding author.

6-2- Funding

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6-3- Conflicts of Interest

The author declare that there is no conflict of interests regarding the publication of this manuscript. In addition, the ethical issues, including plagiarism, informed consent, misconduct, data fabrication and/or falsification, double publication and/or submission, and redundancies have been completely observed by the authors.

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